A New Approach for Reduced Order Modeling of Mechanical Systems Using Vibration Measurements

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Abstract

This study investigates the possibilities of obtaining reduced order mass–damping–stiffness models of mechanical systems using state space realizations identified via dynamic tests. It is shown that even when the system is insufficiently instrumented with sensors and actuators, it is still possible to create physically meaningful reduced order mass–damping–stiffness models that incorporate measured and unmeasured degrees of freedom. It is further discussed that certain assumptions, such as having a diagonal mass matrix or having classical damping in the system, allow one to develop alternative reduced order representations with different physical interpretations. The theoretical presentation is supplemented by a numerical example that illustrates the applications of the formulations developed herein.

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1 Introduction

A well known technique employed in modeling the dynamics of mechanical systems is the use of second order matrix differential equations. In such a formulation, the coefficient matrices contain the physical mass, damping, and stiffness parameters of the system, which in turn affect the modal vibrational parameters such as the natural frequencies and modeshapes. The construction of a mass–damping–stiffness model based on material properties and the system's geometry, as done in finite element analysis, is a relatively straight forward procedure and is widely employed in predicting the response of a structure to prescribed inputs (forward analysis). The identification of such a model from the measured dynamic response, on the other hand, has proven to be a tough challenge, and it is often referred to as the "(linear) inverse vibration problem".

This problem has been addressed by various scholars in the past as evidenced by the works of Agbabian *et al.* [1], Mottershead and Friswell [2], Berman [3], Baruch [4], Beck and Katafygiotis [5], Alvin *et al.* [6, 7], Tseng *et al.* [8, 9], and Balmès [10]. Recently the authors have presented a solution to this problem based on identified state space realizations [11, 12]. This solution has proven to be more flexible and general than the previously available solutions of the problem utilizing state space realizations, and it has been used effectively in estimating the physical parameters of various structural models [13]. On the other hand, even though the requirements on the number of available sensors and actuators for a full order identification has been improved with the aforementioned solution, the question of obtaining reduced order models in the absence of full instrumentation has not been fully investigated yet. A noteworthy exception to this claim is the study by Alvin *et al.* [7] in which the authors have provided a methodology that utilizes undamped second–order frequencies and mass normalized modeshapes to construct reduced order mass, damping, and stiffness matrices. The current study, therefore, is concerned with developing formulations that address the problem of insufficient instrumentation, and attempts at providing new methods based on identified state space models that utilizes both damped and undamped modal information.

The following sections are devoted to the review of the full order modeling problem, effects of insufficient instrumentation, possible reduced order modeling schemes, and alternative formulations that can be developed by considering frequently employed assumptions such as having a diagonal mass matrix and/or a classically damped system. The proposed methodology/solution consists of three well-defined phases. First, a first-order model of the system is determined using the recorded input-output data. Once a first-order model has been determined, the next step is to construct the transformation matrix that relates the arbitrary coordinates of the identified state space model to the set of modal coordinates that are derived via the symmetric eigenvalue problem formulation. The construction of such a transformation matrix has been shown to be possible if there is (at least) one co–located sensor–actuator pair [11, 12]. With the use of this transformation, it will be shown to be possible to evaluate some partitions of the complex eigenvector matrix of the symmetric eigenvalue problem. Utilizing such information, the last step consists of constructing reduced order mass–damping–stiffness models based on the identified complex eigenvalues and partitions of the complex eigenvector matrix. It will be shown in the latter sections that such reduced order models are physically meaningful, and furthermore, that they can be analytically related to the full order matrices. The last section is devoted to the presentation of a numerical example which illustrates the applicability of the solution and the formulations developed in this study.

2 Statement of the Problem

Consider an N degree-of-freedom viscously damped linear structural system, subjected to r external excitations. The equations of motion for such a system can be expressed as:

$$\boldsymbol{\mathcal{M}}\ddot{\mathbf{q}}(t) + \boldsymbol{\mathcal{L}}\dot{\mathbf{q}}(t) + \boldsymbol{\mathcal{K}}\mathbf{q}(t) = \boldsymbol{\mathcal{B}}\mathbf{u}(t) \tag{1}$$

where $\mathbf{q}(t)$ indicates the vector of the generalized nodal displacements, with $(\dot{\ })$ and $(\ddot{\ })$ representing respectively the first and second order derivatives with respect to time. The vector $\mathbf{u}(t)$, of dimension $r \times 1$, is the input vector containing the r external excitations acting on the system, with $\boldsymbol{\mathcal{B}} \in \Re^{N\times r}$ being the input matrix that relates the inputs to the DOFs. The matrices $\boldsymbol{\mathcal{M}} \in \Re^{N\times N}$, $\boldsymbol{\mathcal{L}} \in \Re^{N\times N}$, and $\boldsymbol{\mathcal{K}} \in \Re^{N\times N}$ are the symmetric positive definite mass, damping, and stiffness matrices, respectively. Let us assume that only m output time histories of the structural response are available, so that the measurement vector $\mathbf{y}(t)$, of dimensions $m \times 1$, can be written as:

$$\mathbf{y}(t) = \left[(\boldsymbol{\mathcal{C}}_p\mathbf{q}(t))^T \quad (\boldsymbol{\mathcal{C}}_v\dot{\mathbf{q}}(t))^T \quad (\boldsymbol{\mathcal{C}}_a\ddot{\mathbf{q}}(t))^T \right]^T \tag{2}$$

where the matrices $\mathcal{C}_p$, $\mathcal{C}_v$, and $\mathcal{C}_a$ relate the measurements to positions, velocities, and accelerations, respectively, and the superscript $()^T$ denotes the transpose.

While the cases of a complete set of sensors ($m = N$, [6]) and of a complete set of actuators ($r = N$, [8], [9]) have been previously addressed, the "more general" case of a sufficient number of sensors and actuators ($m + r = N + 1$) with one co-located sensor-actuator pair has only recently been studied by the authors ([11], [12]). In these recent studies the basic assumption is that, at each degree-of-freedom of the system, there is either a sensor or an actuator with at least one degree-of-freedom having a co-located sensor-actuator pair. In the present study, this assumption is removed and the analysis focuses on the case where, still considering a co-located sensor-actuator pair, a sufficient number of sensors and actuators is not available ($m + r < N + 1$) so that there will be degrees of freedom with neither a sensor nor an actuator. This is a common scenario in real life applications where only limited testing and measuring equipment is available. However, even with these limitations, some dynamic characteristics of the structural system can be retrieved and a "reduced" second-order model of the "larger" structural system can still be obtained.

3 Transformation to a First-Order Modal Model

A well known fact from control theory is that it is possible (and, in some cases, convenient) to transform the system of second-order differential equations of motion into a system of first-order differential equations by introducing a state vector $\mathbf{z}(t) = \left[\mathbf{q}(t)^T \ \dot{\mathbf{q}}(t)^T\right]^T$. As discussed in the works of Luş [11] and De Angelis *et al.* [12], the equations of motion (1) and the output equations (2) can be conveniently rewritten as

$$\begin{bmatrix} \mathcal{L} & \mathcal{M} \\ \mathcal{M} & \mathbf{0} \end{bmatrix} \dot{\mathbf{z}}(t) + \begin{bmatrix} \mathcal{K} & \mathbf{0} \\ \mathbf{0} & -\mathcal{M} \end{bmatrix} \mathbf{z}(t) = \begin{bmatrix} \mathcal{B} \\ \mathbf{0} \end{bmatrix} \mathbf{u}(t) \tag{3a}$$

$$\mathbf{y}(t) = [\mathcal{C}_p \quad \mathbf{0}]\, \mathbf{z}(t) \tag{3b}$$

where, for ease of exposition, we have considered only position measurements in the output equation (3b). It should be pointed out, however, that the following results are valid for any type of measurements (positions, velocities, or accelerations), as discussed in [11, 12]. The advantage of rewriting eqs.(1) into eqs.(3) is that now the associated eigenvalue problem in the state space formulation preserves its symmetry, and this yields a great advantage in posing the identification problem, as will be shown in the following formulations. By indicating with $\boldsymbol{\psi}_{N\times 2N} = [\boldsymbol{\psi}_1\ \boldsymbol{\psi}_2\ \ldots\ \boldsymbol{\psi}_{2N}]$ the matrix containing the eigenvectors of the complex eigenvalue problem

$$\left(\lambda_i^2 \boldsymbol{\mathcal{M}} + \lambda_i \boldsymbol{\mathcal{L}} + \boldsymbol{\mathcal{K}}\right) \boldsymbol{\psi}_i = 0 \tag{4}$$

and with $\boldsymbol{\Lambda}_{2N\times 2N}$ the diagonal matrix containing all the complex eigenvalues λ_i ($i = 1, 2, \ldots, 2N$), it is possible to rewrite eqs.(3) in a modal form. Since the eigenvectors $\boldsymbol{\psi}_i$ ($i = 1, 2, ..., 2N$) can be arbitrarily scaled, the scaling choice considered in this study is such that (see Sestieri and Ibrahim [14], Balmès [10])

$$\begin{bmatrix} \boldsymbol{\psi} \\ \boldsymbol{\psi\Lambda} \end{bmatrix}^T \begin{bmatrix} \boldsymbol{\mathcal{L}} & \boldsymbol{\mathcal{M}} \\ \boldsymbol{\mathcal{M}} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{\psi} \\ \boldsymbol{\psi\Lambda} \end{bmatrix} = \mathbf{I} \tag{5a}$$

$$\begin{bmatrix} \boldsymbol{\psi} \\ \boldsymbol{\psi\Lambda} \end{bmatrix}^T \begin{bmatrix} \boldsymbol{\mathcal{K}} & \mathbf{0} \\ \mathbf{0} & -\boldsymbol{\mathcal{M}} \end{bmatrix} \begin{bmatrix} \boldsymbol{\psi} \\ \boldsymbol{\psi\Lambda} \end{bmatrix} = -\boldsymbol{\Lambda} \tag{5b}$$

and with this scaling, for a *proportionally damped system*, the real and imaginary parts of the components of these complex eigenvectors are equal in magnitude. By using the transformation $\mathbf{z}(t) = \left[\boldsymbol{\psi}^T\ (\boldsymbol{\psi\Lambda})^T\right]^T \boldsymbol{\zeta}(t)$, and pre-multiplying eq.(3a) by $\left[\boldsymbol{\psi}^T\ (\boldsymbol{\psi\Lambda})^T\right]$, the equations in modal coordinates can be written as

$$\dot{\boldsymbol{\zeta}}(t) = \boldsymbol{\Lambda\zeta}(t) + \boldsymbol{\psi}^T \boldsymbol{\mathcal{B}}\mathbf{u}(t) \tag{6a}$$

$$\mathbf{y}(t) = \boldsymbol{\mathcal{C}}_p \boldsymbol{\psi\zeta}(t) \tag{6b}$$

4 Determination of a "Reduced" First-Order Model of the System

Here we assume that a state space realization (in some arbitrary basis) of the dynamic system under investigation has been obtained using general input/output data. Such a realization can be expressed as:

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}_C \mathbf{x}(t) + \mathbf{B}_C \mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}_C \mathbf{x}(t) + \mathbf{D}_C \mathbf{u}(t) \end{aligned} \tag{7}$$

where $\mathbf{A}_C \in \Re^{2N\times 2N}$, $\mathbf{B}_C \in \Re^{2N\times r}$, $\mathbf{C}_C \in \Re^{m\times 2N}$, and $\mathbf{D}_C \in \Re^{m\times r}$ are the continuous time system matrices. In this study, an ERA/OKID based approach, as discussed by Juang *et al.* [15, 16] and Luş *et al.* [17, 18], is considered for the identification of the discrete time system matrices of a state space realization, and these discrete time matrices are converted to their continuous time counterparts using the zero order hold assumption. Here we assume that the external excitation is sufficiently rich so that all the vibrational modes of the structure are adequately excited. The advantages of using the ERA/OKID based approach are: 1) no data manipulation (integration or differentiation) is needed, and 2) it has proven to be quite effective in accurately identifying the dynamic characteristics of complex systems using very limited sets of sensors and actuators, even in the presence of noise (Luş [11]). Hence it is very appropriate for the purpose of this study, which is to analyze the case when the number of sensors and actuators available is much smaller than the number of degrees of freedom of the structure.

By considering the transformation $\mathbf{x}(t) = \boldsymbol{\varphi}\boldsymbol{\theta}(t)$, the continuous time system of eqs.(7) can also be written in modal coordinates as

$$\dot{\boldsymbol{\theta}}(t) = \boldsymbol{\Lambda}'\boldsymbol{\theta}(t) + \boldsymbol{\varphi}^{-1}\mathbf{B}_C\mathbf{u}(t) \tag{8a}$$

$$\mathbf{y}(t) = \mathbf{C}_C\boldsymbol{\varphi}\boldsymbol{\theta}(t) \tag{8b}$$

where the matrix $\boldsymbol{\Lambda}'$ contains the $2N$ continuous time eigenvalues of the identified state space model, and $\boldsymbol{\varphi}$, of size $2N \times 2N$, is the matrix of the corresponding eigenvectors. The matrix $\mathbf{D}_C$ has been omitted in eq.(8b) because it is independent of coordinate transformations, and its presence or absence does not in any way alter the development of subsequent results. At this point, it is noteworthy that, since the dimensions of ($\boldsymbol{\varphi}^{-1}\mathbf{B}_C$) are $2N \times r$ and those of ($\mathbf{C}_C\boldsymbol{\varphi}$) are $m \times 2N$, the modal model represented in eqs.(8b) can "only" be used for sensors and actuators placed at locations specified by the experimental setup used in dynamic testing. This limitation will be overcome by the proposed approach, since it will allow us to expand the input/output mapping to include "new" sensor/actuator locations.

4.1 Identifiable partitions of the complex eigenvector matrix

If the first order system of eqs.(7) was identified using data that actually came from the second order model of eq.(1), the models represented by eqs.(6) and (8) are different models of the same system, with the same set of eigenvalues. Therefore, there must be a transformation matrix, $\boldsymbol{\mathcal{T}}$, that relates these two representations, so that:

$$\boldsymbol{\mathcal{T}}^{-1}\boldsymbol{\Lambda}'\boldsymbol{\mathcal{T}} = \boldsymbol{\Lambda} \tag{9a}$$

$$\boldsymbol{\mathcal{T}}^{-1}\boldsymbol{\varphi}^{-1}\mathbf{B}_C = \boldsymbol{\psi}^T\boldsymbol{\mathcal{B}} \tag{9b}$$

$$\mathbf{C}_C\boldsymbol{\varphi}\boldsymbol{\mathcal{T}} = \boldsymbol{\mathcal{C}}_p\boldsymbol{\psi} \tag{9c}$$

The matrix $\boldsymbol{\Lambda}$, which belongs to the state space model of eqs.(6), is always diagonal since it is obtained via a symmetric eigenvalue problem. On the other hand, the matrix $\boldsymbol{\Lambda}'$ of eqs.(8) comes from an asymmetric eigenvalue problem, and it is known that diagonalization is not always guaranteed in an asymmetric eigenvalue problem. In this case, however, since the asymmetric eigenvalue problem is in fact derived from the symmetric eigenvalue problem, there will always exist a set of eigenvectors that will yield $\boldsymbol{\Lambda}' = \boldsymbol{\Lambda}$ (see Appendix A for a proof of this statement). Consequently, the transformation matrix $\boldsymbol{\mathcal{T}}$ is also a diagonal matrix, denoted as $\boldsymbol{\mathcal{T}} = diag(t_1, t_2, \ldots, t_{2N})$. Furthermore, it is assumed that the structure is properly constrained (which generically is the case in a modal testing situation) so that there are no rigid body modes. In addition, the input and output matrices ($\boldsymbol{\mathcal{B}}$ and $\boldsymbol{\mathcal{C}}_p$, respectively) of the finite element model, which contain information about the actuator and sensor locations, are assumed to be known.

The identification of the transformation matrix is contingent upon the existence of (at least) one co–located sensor actuator pair. To briefly summarize the procedure presented by Luş [11] and De Angelis *et al.*, let us assume that there is a co–located sensor–actuator pair at the generic j^{th} DOF. The co–location requirement may be written as

$$\boldsymbol{\mathcal{C}}_p(\text{row corresponding to the } j^{th} \text{ DOF}, :)\boldsymbol{\psi} = (\boldsymbol{\psi}^T\boldsymbol{\mathcal{B}}(:, \text{column corresponding to the } j^{th} \text{ DOF}))^T \tag{10}$$

For ease of presentation, we shall resort to the notation $\mathbf{M}(j,:)$ to denote the row of a generic matrix $\mathbf{M}$ that corresponds to the j^{th} DOF, and we shall denote by $\mathbf{M}(:,j)$ the column of that matrix corresponding to the j^{th} DOF. Hence, the co–location requirement is given by $\boldsymbol{\mathcal{C}}_p(j,:)\boldsymbol{\psi} = (\boldsymbol{\psi}^T\boldsymbol{\mathcal{B}}(:,j))^T$, or, with the use of the transformation equations,

$$\mathbf{C}_C(j,:)\boldsymbol{\varphi}\boldsymbol{\mathcal{T}}^2 = (\boldsymbol{\varphi}^{-1}\mathbf{B}_C(:,j))^T \tag{11}$$

which yields $2N$ equations for the $2N$ unknowns in the diagonal matrix $\boldsymbol{\mathcal{T}}$. It should be noted that the co–location requirement does not need to be satisfied in a strict sense. For example, in the case of a rigid body, such a requirement may be rephrased in alternative forms by properly combining the inputs and outputs. In any case, it is evident that the total number of available sensors and actuators has no bearing on the identifiability of the transformation matrix, rather it is in the determination of the eigenvector matrix ψ that the limitation imposed by the insufficient set of sensors/actuators appears. Some components of the matrix ψ can be determined using the information contained in the input and output matrices. When there is a sensor at the k^{th} DOF, then the k^{th} row of the matrix ψ can be evaluated via eq.(9c), which may be written as

$$\psi(k,:) = \mathbf{C}_C(k,:)\boldsymbol{\varphi}\boldsymbol{\mathcal{T}} \tag{12}$$

On the other hand, if there is an actuator located at the k^{th} DOF, then the k^{th} row of the matrix ψ can be obtained using eq.(9b), i.e.

$$\psi(k,:) = (\boldsymbol{\mathcal{T}}^{-1}\boldsymbol{\varphi}^{-1}\mathbf{B}_C(:,k))^T \tag{13}$$

Clearly, this argument can be applied to determine “only” $n = m + r - 1$ rows of the eigenvector matrix ψ, assuming that there is only one co-located sensor-actuator pair ($n = m + r - n_c$ if there are n_c co-located sensor-actuator pairs). It is important to notice that, in correspondence of a degree of freedom with either a sensor or an actuator, the entire row (corresponding to that degree of freedom) of the eigenvector matrix ψ can be evaluated. This is equivalent to saying that, at each of these degrees of freedom, the corresponding components of all the $2N$ complex vibrational modes are determined. For the other degrees of freedom that are not instrumented with either a sensor or an actuator, however, the corresponding components of the complex eigenvector matrix ψ can not be evaluated since it will not be possible to set up eq.(12) or eq.(13). For a detailed discussion of the aforementioned methodology, the reader is referred to the presentations in the works of Luş [11] and De Angelis *et al.* [12].

Once all the identifiable rows of the matrix ψ have been computed using either eq.(12) or eq.(13), it is convenient to rearrange them by moving all the known rows at the top of the matrix ψ while the unknown rows are moved down. This is equivalent to rearranging the vector of the degrees of freedom in “known” and "unknown" DOFs. The eigenvector matrix $\boldsymbol{\psi}$ can then be represented as:

$$\boldsymbol{\psi} = \begin{bmatrix} \widehat{\boldsymbol{\psi}}_{1,1} & \widehat{\boldsymbol{\psi}}_{1,2} & \cdots & \widehat{\boldsymbol{\psi}}_{1,2N} \\ \widehat{\boldsymbol{\psi}}_{2,1} & \widehat{\boldsymbol{\psi}}_{2,2} & \cdots & \widehat{\boldsymbol{\psi}}_{2,2N} \\ \vdots & \vdots & \vdots & \vdots \\ \widehat{\boldsymbol{\psi}}_{n,1} & \widehat{\boldsymbol{\psi}}_{n,2} & \cdots & \widehat{\boldsymbol{\psi}}_{n,2N} \\ \overline{\boldsymbol{\psi}}_{n+1,1} & \overline{\boldsymbol{\psi}}_{n+1,2} & \cdots & \overline{\boldsymbol{\psi}}_{n+1,2N} \\ \vdots & \vdots & \vdots & \vdots \\ \overline{\boldsymbol{\psi}}_{N,1} & \overline{\boldsymbol{\psi}}_{N,2} & \cdots & \overline{\boldsymbol{\psi}}_{N,2N} \end{bmatrix} \tag{14}$$

where $\widehat{\boldsymbol{\psi}}_{i,j}$ denotes the "*known*" component of the j^{th} mode at the i^{th} degree of freedom, while $\overline{\boldsymbol{\psi}}_{k,j}$ denotes the "*unknown*" component of the j^{th} mode at the k^{th} degree of freedom.

4.2 Expanding the input-output mapping

Having determined the n rows of the matrix $\boldsymbol{\psi}$ (denoted by $\widehat{\boldsymbol{\psi}}$), it is now possible to construct a new state space model, which can also predict the system's response at *actuator* locations for excitations applied at *sensor* locations. This new system is an improvement on the initial first order system, for which the actuator and sensor locations were fixed and limited to the initial test configuration.

If a certain degree of freedom, e.g. the j^{th} DOF, has an actuator placed on it, then the contribution of the excitation to the state equation is through the term $\boldsymbol{\psi}^T\boldsymbol{\mathcal{B}}(:,j)$. Analogously, if a sensor is placed on the j^{th} DOF, then the state vector is related to that output through the term $\boldsymbol{\mathcal{C}}_p(j,:)\boldsymbol{\psi}$. By assuming that sensors are placed at the DOFs where the actuators are placed and that actuators are placed at the DOFs with sensors, a hypothetical "expanded set of $(m + r - n_c)$ co–located sensor–actuator pairs" can be created. For each of such pairs, the co–location requirement is given by $\boldsymbol{\mathcal{C}}_p(j,:) = \boldsymbol{\mathcal{B}}(:,j)^T$, for $j = 1\ to\ (m + r - n_c)$. This would allow us to create new input and output matrices, namely $\widehat{\boldsymbol{\mathcal{B}}}$ and $\widehat{\boldsymbol{\mathcal{C}}}_p$, that can be used in a new state equation, which can be written as

$$\dot{\boldsymbol{\zeta}}(t) = \boldsymbol{\Lambda}\boldsymbol{\zeta}(t) + \widehat{\boldsymbol{\psi}}^T\widehat{\boldsymbol{\mathcal{B}}}\widehat{\mathbf{u}}(t) \tag{15a}$$

$$\widehat{\mathbf{y}}(t) = \widehat{\boldsymbol{\mathcal{C}}}_p\widehat{\boldsymbol{\psi}}\boldsymbol{\zeta}(t) \tag{15b}$$

where the quantities with $(\widehat{\ })$ are related to the n degrees of freedom with either a sensor or an actuator. It is important to note that the new input vector $\widehat{\mathbf{u}}(t)$ and the new output vector $\widehat{\mathbf{y}}(t)$ now contain information about all the n active degrees of freedom, providing a more general input/output mapping. This is to say that 1) it is now possible to predict the output at any of the DOFs that did not have a sensor but had an actuator placed on them, and 2) if it so happens that the system is subjected to a new input applied at any of the aforementioned n active DOFs, it is also possible to accurately predict the response at any of those locations. This operation can be seen as the time domain equivalent of building the transfer function matrix of the system using the symmetry property of such a matrix. If, for example, one had identified the components of the transfer function matrix that relates the i^{th} input to the j^{th} output, then it is also possible to obtain immediately the component relating a hypothetical output at the i^{th} DOF to a hypothetical input at the j^{th} DOF.

5 Retrieving the Mass, Damping and Stiffness Matrices of the "Reduced Order Model"

5.1 The General Case

Having determined the n rows of the matrix $\boldsymbol{\psi}$ (denoted by $\widehat{\boldsymbol{\psi}}$), it is now possible to determine a compact form of the mass, damping and stiffness matrices related to the reduced model. To this end, let us consider the general expressions of the mass, damping and stiffness matrices of the larger system which are obtained by imposing the orthogonality conditions given in eq.(5). As shown in [10] and [12], these matrices can be expressed as:

$$\boldsymbol{\mathcal{M}} = (\boldsymbol{\psi}\boldsymbol{\Lambda}\boldsymbol{\psi}^T)^{-1} \tag{16a}$$

$$\boldsymbol{\mathcal{L}} = -\boldsymbol{\mathcal{M}}\boldsymbol{\psi}\boldsymbol{\Lambda}^2\boldsymbol{\psi}^T\boldsymbol{\mathcal{M}} \tag{16b}$$

$$\boldsymbol{\mathcal{K}} = -(\boldsymbol{\psi}\boldsymbol{\Lambda}^{-1}\boldsymbol{\psi}^T)^{-1} \tag{16c}$$

Since the mass and stiffness matrices have similar structures, let us first analyze the reduced form of the mass and stiffness matrices. Using the subdivision of the matrix $\boldsymbol{\psi}$ presented in eq.(14), it is possible to express the matrices $\boldsymbol{\mathcal{M}}$ and $\boldsymbol{\mathcal{K}}$ in partitioned forms as:

$$\boldsymbol{\mathcal{M}} = \begin{bmatrix} \widehat{\boldsymbol{\psi}}\boldsymbol{\Lambda}\widehat{\boldsymbol{\psi}}^T & \widehat{\boldsymbol{\psi}}\boldsymbol{\Lambda}\overline{\boldsymbol{\psi}}^T \\ \overline{\boldsymbol{\psi}}\boldsymbol{\Lambda}\widehat{\boldsymbol{\psi}}^T & \overline{\boldsymbol{\psi}}\boldsymbol{\Lambda}\overline{\boldsymbol{\psi}}^T \end{bmatrix}^{-1} \; ; \; \boldsymbol{\mathcal{K}} = -\begin{bmatrix} \widehat{\boldsymbol{\psi}}\boldsymbol{\Lambda}^{-1}\widehat{\boldsymbol{\psi}}^T & \widehat{\boldsymbol{\psi}}\boldsymbol{\Lambda}^{-1}\overline{\boldsymbol{\psi}}^T \\ \overline{\boldsymbol{\psi}}\boldsymbol{\Lambda}^{-1}\widehat{\boldsymbol{\psi}}^T & \overline{\boldsymbol{\psi}}\boldsymbol{\Lambda}^{-1}\overline{\boldsymbol{\psi}}^T \end{bmatrix}^{-1} \tag{17}$$

where $\widehat{\boldsymbol{\psi}}$, of dimension $n \times 2N$, and $\overline{\boldsymbol{\psi}}$, of dimension $(N-n)\times 2N$, are the submatrices of $\boldsymbol{\psi}$ corresponding to known and unknown degrees of freedom, respectively. By employing the inverse of only the known portion of $\boldsymbol{\mathcal{M}}^{-1}$, it is possible to obtain a "reduced" order mass matrix of the structural system, $\widehat{\boldsymbol{\mathcal{M}}}$, of dimension $n \times n$, as:

$$\widehat{\boldsymbol{\mathcal{M}}} = [\widehat{\boldsymbol{\psi}}\boldsymbol{\Lambda}\widehat{\boldsymbol{\psi}}^T]^{-1} \tag{18}$$

Similarly, a "reduced" order $n \times n$ stiffness matrix of the system, $\widehat{\boldsymbol{\mathcal{K}}}$,can be obtained as:

$$\widehat{\boldsymbol{\mathcal{K}}} = -[\widehat{\boldsymbol{\psi}}\boldsymbol{\Lambda}^{-1}\widehat{\boldsymbol{\psi}}^T]^{-1} \tag{19}$$

These two matrices are symmetric and are related to the general $N \times N$ mass and stiffness matrices of the structural system through a static condensation relationship. In fact, from eq.(18) and (19), both matrices are presented as the inverse of a symmetric matrix which itself is a partition of a larger matrix. If we now rewrite the partitioned form of $\boldsymbol{\mathcal{K}}$ as:

$$\boldsymbol{\mathcal{K}} = \begin{bmatrix} \boldsymbol{\mathcal{K}}_{nn} & \boldsymbol{\mathcal{K}}_{nu} \\ \boldsymbol{\mathcal{K}}_{un} & \boldsymbol{\mathcal{K}}_{uu} \end{bmatrix} = -\begin{bmatrix} \widehat{\boldsymbol{\psi}}\boldsymbol{\Lambda}^{-1}\widehat{\boldsymbol{\psi}}^T & \widehat{\boldsymbol{\psi}}\boldsymbol{\Lambda}^{-1}\overline{\boldsymbol{\psi}}^T \\ \overline{\boldsymbol{\psi}}\boldsymbol{\Lambda}^{-1}\widehat{\boldsymbol{\psi}}^T & \overline{\boldsymbol{\psi}}\boldsymbol{\Lambda}^{-1}\overline{\boldsymbol{\psi}}^T \end{bmatrix}^{-1} \tag{20}$$

then what we have denoted as $\widehat{\boldsymbol{\mathcal{K}}} = -[\widehat{\boldsymbol{\psi}}\boldsymbol{\Lambda}^{-1}\widehat{\boldsymbol{\psi}}^T]^{-1}$ is in fact

$$\widehat{\boldsymbol{\mathcal{K}}} = [\boldsymbol{\mathcal{K}}_{nn} - \boldsymbol{\mathcal{K}}_{nu}\boldsymbol{\mathcal{K}}_{uu}^{-1}\boldsymbol{\mathcal{K}}_{un}] \tag{21}$$

It is important to note that the expression in eq.(21) is identical to an expression we would have got if we had considered the static condensation of the matrix $\boldsymbol{\mathcal{K}}$ by employing the DOFs with either an actuator or a sensor as the "independent" DOFs, and the DOFs with neither an actuator nor a sensor as "dependent" (condensed) DOFs. This expression of $\widehat{\boldsymbol{\mathcal{K}}}$ is also identical to the reduced stiffness matrix one would obtain using the Guyan reduction [19]. Because of the similarity in the structure of $\boldsymbol{\mathcal{M}}$ and $\boldsymbol{\mathcal{K}}$, the first argument holds true also for the mass matrix, so that the "reduced" mass matrix corresponds to

$$\widehat{\boldsymbol{\mathcal{M}}} = [\boldsymbol{\mathcal{M}}_{nn} - \boldsymbol{\mathcal{M}}_{nu}\boldsymbol{\mathcal{M}}_{uu}^{-1}\boldsymbol{\mathcal{M}}_{un}] \tag{22}$$

with the partitions defined analogously. However, since this reduced mass matrix is obtained only with the partitions of the larger mass matrix and does not involve any contribution from the stiffness matrix, it is in general quite different from the reduced mass matrix one would get using the Guyan reduction. In fact, these reduced order matrices $\widehat{\mathcal{K}}$ and $\widehat{\mathcal{M}}$ are also known as the Schur complements of the partitions $\mathcal{K}_{uu}$ and $\mathcal{M}_{uu}$, respectively.

With regard to the damping matrix, its reduced form can be obtained from eq.(16), by accounting only for the known partitions of ψ and $\mathcal{M}$. This reduced order damping matrix $\widehat{\mathcal{L}}$, of dimensions $n \times n$, is symmetric and can be expressed as:

$$\widehat{\mathcal{L}} = -\widehat{\mathcal{M}}\widehat{\psi}\boldsymbol{\Lambda}^2\widehat{\psi}^T\widehat{\mathcal{M}} \tag{23}$$

where the information about all the N DOFs of the "larger" system have been included through $\widehat{\mathcal{M}}$, $\widehat{\psi}$, and $\boldsymbol{\Lambda}$.

5.2 Block Diagonal Mass Matrix Case

If the system under consideration has a block diagonal mass matrix, it is possible to provide new interpretations to the results of the previous section. Let us first note that a block diagonal mass matrix may be partitioned as

$$\mathcal{M} = \begin{bmatrix} \mathcal{M}_{nn} & \mathbf{0} \\ \mathbf{0} & \mathcal{M}_{uu} \end{bmatrix}$$

and that for this case eq.(22) leads to

$$\widehat{\mathcal{M}} = [\widehat{\psi}\boldsymbol{\Lambda}\widehat{\psi}^T]^{-1} = \mathcal{M}_{nn} \tag{24}$$

Therefore, for a system with a block diagonal mass matrix, the subpartition of the mass matrix related to the instrumented DOFs can be directly and exactly evaluated using the available complex modal information. Similarly, since in this case the expression for the damping matrix can be written as

$$\mathcal{L} = \begin{bmatrix} \mathcal{L}_{nn} & \mathcal{L}_{nu} \\ \mathcal{L}_{un} & \mathcal{L}_{uu} \end{bmatrix} = -\begin{bmatrix} \mathcal{M}_{nn} & \mathbf{0} \\ \mathbf{0} & \mathcal{M}_{uu} \end{bmatrix} \begin{bmatrix} \widehat{\psi}\boldsymbol{\Lambda}^2\widehat{\psi}^T & \widehat{\psi}\boldsymbol{\Lambda}^2\overline{\psi}^T \\ \overline{\psi}\boldsymbol{\Lambda}^{-2}\widehat{\psi}^T & \overline{\psi}\boldsymbol{\Lambda}^2\overline{\psi}^T \end{bmatrix} \begin{bmatrix} \mathcal{M}_{nn} & \mathbf{0} \\ \mathbf{0} & \mathcal{M}_{uu} \end{bmatrix} \tag{25}$$

the partition $\mathcal{L}_{nn}$ can also be directly and exactly evaluated as

$$\mathcal{L}_{nn} = -\mathcal{M}_{nn}\widehat{\psi}\boldsymbol{\Lambda}^2\widehat{\psi}^T\mathcal{M}_{nn} \tag{26}$$

The partitions of the stiffness matrix, however, are not so straight forward to evaluate, and only approximate results may be obtained. In order to clarify this comment, let us go back to eq.(5b) and rewrite it as

$$\boldsymbol{\psi}^T \boldsymbol{\mathcal{K}} \boldsymbol{\psi} - \boldsymbol{\Lambda} \boldsymbol{\psi}^T \boldsymbol{\mathcal{M}} \boldsymbol{\psi} \boldsymbol{\Lambda} = -\boldsymbol{\Lambda} \tag{27}$$

Pre–multiplying eq.(27) with $\boldsymbol{\psi}\boldsymbol{\Lambda}$ and post–multiplying it with $\boldsymbol{\Lambda}\boldsymbol{\psi}^T$ leads, with the help of the relations in eq.(16), to

$$\boldsymbol{\mathcal{K}} = -\boldsymbol{\mathcal{M}} \boldsymbol{\psi} \boldsymbol{\Lambda}^3 \boldsymbol{\psi}^T \boldsymbol{\mathcal{M}} + \boldsymbol{\mathcal{L}} \boldsymbol{\mathcal{M}}^{-1} \boldsymbol{\mathcal{L}} \tag{28}$$

and, using the partitions of the mass and damping matrices, eq.(28) may be rewritten as

$$\boldsymbol{\mathcal{K}}_{nn} = \boldsymbol{\mathcal{L}}_{nn} \boldsymbol{\mathcal{M}}_{nn}^{-1} \boldsymbol{\mathcal{L}}_{nn} + \boldsymbol{\mathcal{L}}_{nu} \boldsymbol{\mathcal{M}}_{uu}^{-1} \boldsymbol{\mathcal{L}}_{un} - \boldsymbol{\mathcal{M}}_{nn} \widehat{\boldsymbol{\psi}} \boldsymbol{\Lambda}^3 \widehat{\boldsymbol{\psi}}^T \boldsymbol{\mathcal{M}}_{nn} \tag{29}$$

Clearly this expression can not be evaluated exactly since $\boldsymbol{\mathcal{L}}_{nu}$, $\boldsymbol{\mathcal{M}}_{uu}$, and $\boldsymbol{\mathcal{L}}_{un}$ are unknown quantities. The "closeness" of this estimation naturally depends on the contribution of the unknown term, and unfortunately it is not trivial to quantify due to its sole dependence on unattainable parameters of the system.

5.3 Diagonal Mass Matrix Case with Mass Normalized Normal Modes

The discussion until now has focused on the complex eigenvectors and eigenvalues which are a direct product of the state space formulation. The normal modal parameters, on the other hand, are also widely used in modal analysis, and in this section we present a new methodology that utilizes such information to construct partitions of the mass, damping, and stiffness matrices.

The eigenvalue problem for the so called normal modal parameters is given by

$$\boldsymbol{\mathcal{M}} \boldsymbol{\phi} \boldsymbol{\Omega}^2 = \boldsymbol{\mathcal{K}} \boldsymbol{\phi} \tag{30}$$

where $\boldsymbol{\Omega}^2 = diag(\Omega_1^2, \Omega_2^2, \ldots, \Omega_N^2)$ is a diagonal matrix containing the squares of the undamped natural frequencies, and $\boldsymbol{\phi}_{N\times N}$ is the eigenvector matrix whose columns are the normal (undamped) eigenvectors of the system. Here we assume that the eigenvectors are mass normalized, i.e. that they are scaled such that

$$\boldsymbol{\phi}^T \boldsymbol{\mathcal{M}} \boldsymbol{\phi} = \mathbf{I}, \quad \boldsymbol{\phi}^T \boldsymbol{\mathcal{K}} \boldsymbol{\phi} = \boldsymbol{\Omega}^2 \tag{31}$$

The determination of these normal modal parameters from experiments is possible for classically damped systems; the reader is referred to the work of Alvin *et al.* [7] for a brief discussion of candidate procedures. Here we employ the same assumptions as Alvin *et al.* [7] regarding the availability of these modal parameters, and assume that (i) the system is classically damped, so that the undamped eigenvalues and modal damping percentages can be easily evaluated from the continuous time poles of the identified state space model (as discussed, for example, by Luş [11]), and that (ii) the mass normalized undamped mode shapes are known at the sensor locations.

Using these normal modal parameters, the mass, damping, and stiffness matrices can be constructed using

$$\boldsymbol{\mathcal{M}} = \boldsymbol{\phi}^{-T}\boldsymbol{\phi}^{-1}, \quad \boldsymbol{\mathcal{L}} = \boldsymbol{\phi}^{-T}\boldsymbol{\mathcal{E}}\boldsymbol{\phi}^{-1}, \quad \boldsymbol{\mathcal{K}} = \boldsymbol{\phi}^{-T}\boldsymbol{\Omega}^2\boldsymbol{\phi}^{-1} \tag{32}$$

where $\boldsymbol{\mathcal{E}}$ is the damping matrix in modal coordinates (diagonal for a classically damped system). If, on the other hand, the system is not fully instrumented, then it will not be possible to evaluate the whole eigenvector matrix $\boldsymbol{\phi}$, and hence eqs.(32) will not be applicable. It will still be possible, however, to employ alternate expressions to estimate partitions of the mass, damping, and stiffness matrices, provided that the system has a diagonal mass matrix. To investigate this claim, let us start by partitioning the eigenvector matrix such that

$$\boldsymbol{\phi} = \begin{bmatrix} \widehat{\boldsymbol{\phi}} \\ \overline{\boldsymbol{\phi}} \end{bmatrix} \tag{33}$$

where $\widehat{\boldsymbol{\phi}}_{n\times N}$ is the partition of $\boldsymbol{\phi}$ that can be determined, and $\overline{\boldsymbol{\phi}}_{(N-n)\times N}$ denotes the partition that can not be evaluated due to insufficient instrumentation. Furthermore, the singular value decomposition of $\boldsymbol{\phi}$ is given by

$$\boldsymbol{\phi} = \begin{bmatrix} \widehat{\boldsymbol{\phi}} \\ \overline{\boldsymbol{\phi}} \end{bmatrix} = \mathbf{U}\mathbf{S}\mathbf{V}^T = \begin{bmatrix} \widehat{\mathbf{U}}\mathbf{S}\mathbf{V}^T \\ \overline{\mathbf{U}}\mathbf{S}\mathbf{V}^T \end{bmatrix} \tag{34}$$

where $\mathbf{U}_{N\times N}$ and $\mathbf{V}_{N\times N}$ are real unitary matrices containing the left and right singular vectors, and $\mathbf{S}_{N\times N}$ is a diagonal matrix containing the singular values. Using this decomposition, the inverse of $\boldsymbol{\phi}$ can be expressed as

$$\boldsymbol{\phi}^{-1} = \mathbf{V}\mathbf{S}^{-1}\mathbf{U}^T = \begin{bmatrix} \mathbf{V}\mathbf{S}^{-1}\widehat{\mathbf{U}}^T & \mathbf{V}\mathbf{S}^{-1}\overline{\mathbf{U}}^T \end{bmatrix} \tag{35}$$

and, using eqs.(32), the expressions for the mass, damping, and stiffness matrices can be written as

$$\boldsymbol{\mathcal{M}} = \begin{bmatrix} \boldsymbol{\mathcal{M}}_{nn} & \boldsymbol{\mathcal{M}}_{nu} \\ \boldsymbol{\mathcal{M}}_{un} & \boldsymbol{\mathcal{M}}_{uu} \end{bmatrix} = \begin{bmatrix} \widehat{\mathbf{U}}\mathbf{S}^{-1}\mathbf{V}^T\mathbf{V}\mathbf{S}\widehat{\mathbf{U}}^T & \widehat{\mathbf{U}}\mathbf{S}^{-1}\mathbf{V}^T\mathbf{V}\mathbf{S}\overline{\mathbf{U}}^T \\ \overline{\mathbf{U}}\mathbf{S}^{-1}\mathbf{V}^T\mathbf{V}\mathbf{S}\widehat{\mathbf{U}}^T & \overline{\mathbf{U}}\mathbf{S}^{-1}\mathbf{V}^T\mathbf{V}\mathbf{S}\overline{\mathbf{U}}^T \end{bmatrix} \tag{36a}$$

$$\boldsymbol{\mathcal{L}} = \begin{bmatrix} \boldsymbol{\mathcal{L}}_{nn} & \boldsymbol{\mathcal{L}}_{nu} \\ \boldsymbol{\mathcal{L}}_{un} & \boldsymbol{\mathcal{L}}_{uu} \end{bmatrix} = \begin{bmatrix} \widehat{\mathbf{U}}\mathbf{S}^{-1}\mathbf{V}^T\boldsymbol{\mathcal{E}}\mathbf{V}\mathbf{S}\widehat{\mathbf{U}}^T & \widehat{\mathbf{U}}\mathbf{S}^{-1}\mathbf{V}^T\boldsymbol{\mathcal{E}}\mathbf{V}\mathbf{S}\overline{\mathbf{U}}^T \\ \overline{\mathbf{U}}\mathbf{S}^{-1}\mathbf{V}^T\boldsymbol{\mathcal{E}}\mathbf{V}\mathbf{S}\widehat{\mathbf{U}}^T & \overline{\mathbf{U}}\mathbf{S}^{-1}\mathbf{V}^T\boldsymbol{\mathcal{E}}\mathbf{V}\mathbf{S}\overline{\mathbf{U}}^T \end{bmatrix} \tag{36b}$$

$$\boldsymbol{\mathcal{K}} = \begin{bmatrix} \boldsymbol{\mathcal{K}}_{nn} & \boldsymbol{\mathcal{K}}_{nu} \\ \boldsymbol{\mathcal{K}}_{un} & \boldsymbol{\mathcal{K}}_{uu} \end{bmatrix} = \begin{bmatrix} \widehat{\mathbf{U}}\mathbf{S}^{-1}\mathbf{V}^T\boldsymbol{\Omega}^2\mathbf{V}\mathbf{S}\widehat{\mathbf{U}}^T & \widehat{\mathbf{U}}\mathbf{S}^{-1}\mathbf{V}^T\boldsymbol{\Omega}^2\mathbf{V}\mathbf{S}\overline{\mathbf{U}}^T \\ \overline{\mathbf{U}}\mathbf{S}^{-1}\mathbf{V}^T\boldsymbol{\Omega}^2\mathbf{V}\mathbf{S}\widehat{\mathbf{U}}^T & \overline{\mathbf{U}}\mathbf{S}^{-1}\mathbf{V}^T\boldsymbol{\Omega}^2\mathbf{V}\mathbf{S}\overline{\mathbf{U}}^T \end{bmatrix} \tag{36c}$$

It should be emphasized that, since the whole eigenvector matrix $\boldsymbol{\phi}$ is not available, it is in general not possible to determine the matrices $\widehat{\mathbf{U}}$, $\overline{\mathbf{U}}$, $\mathbf{S}$, and $\mathbf{V}$. It can be shown, however, that if the system has a diagonal mass matrix, then the partition $\mathbf{V}\mathbf{S}^{-1}\widehat{\mathbf{U}}^T$ is identically equal to the Moore–Penrose pseudo–inverse of $\widehat{\boldsymbol{\phi}}$, i.e.

$$\mathbf{V}\mathbf{S}^{-1}\widehat{\mathbf{U}}^T = \widehat{\boldsymbol{\phi}}^{\dagger} \tag{37}$$

where $\widehat{\boldsymbol{\phi}}^{\dagger}$ denotes the Moore–Penrose pseudo–inverse (See Appendix B for the proof of the statement). Therefore, the partitions of the mass, damping, and stiffness matrices corresponding to the known DOFs can be exactly evaluated using

$$\boldsymbol{\mathcal{M}}_{nn} = (\widehat{\boldsymbol{\phi}}^{\dagger})^T(\widehat{\boldsymbol{\phi}}^{\dagger}), \quad \boldsymbol{\mathcal{L}}_{nn} = (\widehat{\boldsymbol{\phi}}^{\dagger})^T\boldsymbol{\mathcal{E}}(\widehat{\boldsymbol{\phi}}^{\dagger}), \quad \boldsymbol{\mathcal{K}}_{nn} = (\widehat{\boldsymbol{\phi}}^{\dagger})^T\boldsymbol{\Omega}^2(\widehat{\boldsymbol{\phi}}^{\dagger}) \tag{38}$$

It should be noted that these expressions complement the results presented in the work of Alvin *et al.* [7], wherein the authors had provided expressions that yielded reduced order matrices equivalent to ones that would be obtained via Guyan reduction.

As a final note, it should be mentioned that reduced order physical matrices, obtained via the procedures described herein, do not completely reflect the full dynamics of the system, i.e. the response of the measured/excited DOFs as predicted by the reduced order system will not be equal to the true response of the respective DOFs of the full order model. This problem is common to all reduction schemes, whether they start from an identified model or an analytical model, and is mainly due to the fact that the reduced order matrices lead to a different eigenvalue problem than that of the full order matrices, and that modal properties of the full order system are not preserved in this new eigenvalue problem. Therefore it is best to use the information obtained via the proposed schemes for obtaining reduced order models in investigating the partitions of the full order matrices, e.g. in health monitoring or model updating. In order to accurately predict the structural response, one should use the state space model represented by the system in eqs.(15).

6 Numerical Example

In order to present the various stages of the proposed approach, let us consider a brief numerical example. The system we study is the 4 DOF lumped mass model shown in Figure 1. The dynamic data to be used in the identification is obtained by subjecting this system to a white noise input at the second DOF, and displacements are assumed to be measured at the first and the second DOFs. It should be strongly emphasized, however, that the methodology is also applicable with velocity and/or acceleration measurements. The reader is referred to the works of Luş [11] and De Angelis *et al.* [12] for a through discussion of this issue.

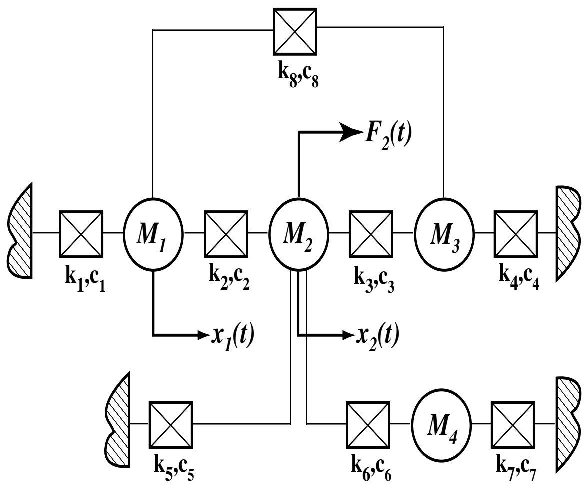

Figure 1: 4 DOF lumped mass system

The system properties are chosen as follows: $M_1 = 0.8$, $M_2 = 2.0$, $M_3 = 1.2$, $M_4 = 0.6$, $k_i = 1.0$ for $i = 1, 4, 5, 7$, $k_i = 2.0$ for $i = 2, 3, 6, 8$, and $c_i = 0.1 \times k_i$ for all i. Note that this choice of the viscous damping coefficients leads to a classically damped system, thereby allowing us to discuss all the formulations developed in this study. With these parameters, the mass, damping, and stiffness matrices can be constructed as

$$\boldsymbol{\mathcal{M}} = \begin{bmatrix} 0.8 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1.2 & 0 \\ 0 & 0 & 0 & 0.6 \end{bmatrix} ; \quad \boldsymbol{\mathcal{L}} = \begin{bmatrix} 0.4 & -0.1 & -0.1 & 0 \\ -0.1 & 0.5 & -0.1 & -0.1 \\ -0.1 & -0.1 & 0.4 & 0 \\ 0 & -0.1 & 0 & 0.3 \end{bmatrix} ; \quad \boldsymbol{\mathcal{K}} = \begin{bmatrix} 4 & -1 & -1 & 0 \\ -1 & 5 & -1 & -1 \\ -1 & -1 & 4 & 0 \\ 0 & -1 & 0 & 3 \end{bmatrix}$$

Note that the partitions of the stiffness matrix defined in eq.(20) are given for this system and instrumentation set up by

$$\boldsymbol{\mathcal{K}}_{nn} = \begin{bmatrix} 4 & -1 \\ -1 & 5 \end{bmatrix} ; \quad \boldsymbol{\mathcal{K}}_{nu} = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}$$

$$\boldsymbol{\mathcal{K}}_{un} = \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix} ; \quad \boldsymbol{\mathcal{K}}_{uu} = \begin{bmatrix} -4 & 0 \\ 0 & 3 \end{bmatrix}$$

The partitions of the mass and the damping matrices are defined analogously.

The initial stage in the proposed methodology is the identification of a first order model of the system. Using the OKID/ERA approach [16, 18], a single input - two output system is easily identified. The success of this identification may be easily judged by comparing the actual and identified values of the continuous time poles, which are presented in Table 1.

Clearly, the OKID/ERA algorithm has been extremely successful in realizing an initial state space model even with two outputs and a single input, as evidenced by the exact agreement between the actual and identified values of the poles. Also note that with the rich input employed in the test, it was in fact possible to identify all the poles of the system. Starting with this initial model, the next step is to find the matrix $\boldsymbol{\mathcal{T}}$ that allows us to transform the equations in eqs.(8) to the modal coordinates of the symmetric eigenvalue problem given by eqs.(6). Since the system has a co–located sensor–actuator pair located at the second DOF, it is indeed possible to identify the transformation matrix and execute the transformation. Furthermore, using the equations in the transformed modal coordinates, the partitions of the complex eigenvector matrix corresponding to the instrumented DOFs (the first and the second DOFs for this example) may be evaluated. In fact, in this case we get $\widehat{\boldsymbol{\psi}} = \left[\widehat{\boldsymbol{\psi}}_1 \; \widehat{\boldsymbol{\psi}}_1^* \; \widehat{\boldsymbol{\psi}}_2 \; \widehat{\boldsymbol{\psi}}_2^* \; \widehat{\boldsymbol{\psi}}_3 \; \widehat{\boldsymbol{\psi}}_3^* \; \widehat{\boldsymbol{\psi}}_4 \; \widehat{\boldsymbol{\psi}}_4^* \right]$, where the superscript $(\;)^*$ denotes complex

Actual		Identified	
$\Re(\lambda_i)$	$\Im(\lambda_i)$	$\Re(\lambda_i)$	$\Im(\lambda_i)$
-0.077	-1.244	-0.077	-1.244
-0.077	+1.244	-0.077	+1.244
-0.170	-1.838	-0.170	-1.838
-0.170	+1.838	-0.170	+1.838
-0.263	-2.278	-0.263	-2.278
-0.263	+2.278	-0.263	+2.278
-0.281	-2.353	-0.281	-2.353
-0.281	+2.353	-0.281	+2.353

Table 1: Actual and identified continuous time poles of the model. $\Re()$ and $\Im()$ refer, respectively, to the real and imaginary components of the poles.

conjugate, with

$$\widehat{\boldsymbol{\psi}}_1 = \begin{bmatrix} 0.1613 + \jmath 0.1613 \\ 0.2515 + \jmath 0.2515 \end{bmatrix}; \quad \widehat{\boldsymbol{\psi}}_2 = \begin{bmatrix} 0.1054 + \jmath 0.1054 \\ -0.1288 - \jmath 0.1288 \end{bmatrix}$$

$$\widehat{\boldsymbol{\psi}}_3 = \begin{bmatrix} 0.1394 + \jmath 0.1394 \\ 0.0560 + \jmath 0.0560 \end{bmatrix}; \quad \widehat{\boldsymbol{\psi}}_4 = \begin{bmatrix} -0.3026 - \jmath 0.3026 \\ 0.0608 + \jmath 0.0608 \end{bmatrix}$$

Note that the real and imaginary parts of the components of these eigenvectors are equal in magnitude; this phenomena is due to the particular scaling choice expressed in eqs.(5) and it is indicative of the fact that the system is classically damped. At this point, we have all the information we need to evaluate the reduced order matrices. Starting with eqs.(18) and (19), we have

$$\widehat{\boldsymbol{\mathcal{M}}} = [\widehat{\boldsymbol{\psi}} \boldsymbol{\Lambda} \widehat{\boldsymbol{\psi}}^T]^{-1} = \begin{bmatrix} 0.8 & 0.0 \\ 0.0 & 2.0 \end{bmatrix} \tag{39a}$$

$$\widehat{\boldsymbol{\mathcal{K}}} = -[\widehat{\boldsymbol{\psi}} \boldsymbol{\Lambda}^{-1} \widehat{\boldsymbol{\psi}}^T]^{-1} = \begin{bmatrix} 3.75 & -1.25 \\ -1.25 & 4.12 \end{bmatrix} \tag{39b}$$

Note that, since the mass matrix was diagonal, the identified reduced order mass matrix $\widehat{\boldsymbol{\mathcal{M}}}$ is nothing but the 2×2 partition corresponding to the first and the second DOFs (i.e. $\boldsymbol{\mathcal{M}}_{nn}$), and this result was to be expected due to eq.(24). The identified reduced order stiffness matrix $\widehat{\boldsymbol{\mathcal{K}}}$, on the other hand, is exactly equal to the matrix one would get by statically condensing the third and the fourth DOFs (i.e. Guyan reduction); the result of such a reduction would be given by the formula $\boldsymbol{\mathcal{K}}_{nn} - \boldsymbol{\mathcal{K}}_{nu}\boldsymbol{\mathcal{K}}_{uu}^{-1}\boldsymbol{\mathcal{K}}_{un}$ with the partitions presented above. Analogously, a reduced order damping matrix can be obtained via eq.(23) as

$$\widehat{\boldsymbol{\mathcal{L}}} = -\widehat{\boldsymbol{\mathcal{M}}}\widehat{\boldsymbol{\psi}}\boldsymbol{\Lambda}^2\widehat{\boldsymbol{\psi}}^T\widehat{\boldsymbol{\mathcal{M}}} = \begin{bmatrix} 0.4 & -0.1 \\ -0.1 & 0.5 \end{bmatrix}$$

Once again, owing to the diagonal nature of the mass matrix, this reduced order damping matrix is nothing but the 2×2 partition corresponding to the first and the second DOFs, i.e. $\widehat{\boldsymbol{\mathcal{L}}} = \boldsymbol{\mathcal{L}}_{nn}$, as was expressed in eq.(26).

An alternate expression one could obtain for a reduced order stiffness matrix would be with the use of eq.(29). In fact, using this equation, one could get an estimate of the partition $\boldsymbol{\mathcal{K}}_{nn}$ by ignoring the contribution of the unknown term $\boldsymbol{\mathcal{L}}_{nu}\boldsymbol{\mathcal{M}}_{uu}^{-1}\boldsymbol{\mathcal{L}}_{un}$, which in this case leads to

$$\boldsymbol{\mathcal{K}}_{nn}^{est} = \begin{bmatrix} 3.99 & -1.01 \\ -1.01 & 4.98 \end{bmatrix}$$

Even though this estimate is in error, it is interesting to note that the error is in fact quite small for the system under consideration. In general, the "closeness" of this estimate will naturally depend on the nature of the neglected term as discussed in the previous sections.

As a final note, let us consider the estimates we could obtain using normal modal parameters. With the instrumentation at hand, it would be possible (see Alvin *et al.* [7] and Bernal and Gunes [20]) to get the partition $\widehat{\boldsymbol{\phi}}$ of the mass normalized eigenvectors as

$$\widehat{\boldsymbol{\phi}} = \begin{bmatrix} -0.3598 & 0.2858 & -0.4207 & 0.9284 \\ -0.5610 & -0.3492 & -0.1691 & -0.1865 \end{bmatrix}$$

Since the second order eigenvalues and modal damping ratios can be evaluated directly from the continuous time poles of the identified first order system (for an explanation of the procedure, see, for example, the study be Luş *et al.* [18]), the second order matrices $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\Omega}^2$ may be easily constructed as

$$\boldsymbol{\Omega}^2 = \begin{bmatrix} 1.553 & 0 & 0 & 0 \\ 0 & 3.407 & 0 & 0 \\ 0 & 0 & 5.257 & 0 \\ 0 & 0 & 0 & 5.616 \end{bmatrix} ; \quad \boldsymbol{\mathcal{E}} = \begin{bmatrix} 0.1553 & 0 & 0 & 0 \\ 0 & 0.3407 & 0 & 0 \\ 0 & 0 & 0.5257 & 0 \\ 0 & 0 & 0 & 0.5616 \end{bmatrix}$$

Hence, using eqs.(38), the partitions $\boldsymbol{\mathcal{M}}_{nn}$, $\boldsymbol{\mathcal{L}}_{nn}$, and $\boldsymbol{\mathcal{K}}_{nn}$ can be evaluated as

$$\boldsymbol{\mathcal{M}}_{nn} = \begin{bmatrix} 0.8 & 0.0 \\ 0.0 & 2.0 \end{bmatrix} ; \quad \boldsymbol{\mathcal{L}}_{nn} = \begin{bmatrix} 0.4 & -0.1 \\ -0.1 & 0.5 \end{bmatrix} ; \quad \boldsymbol{\mathcal{K}}_{nn} = \begin{bmatrix} 4.0 & -1.0 \\ -1.0 & 5.0 \end{bmatrix}$$

and clearly these identified matrices are identically equal to their respective actual values.

7 Conclusions

In this study, the authors have derived various formulations regarding the construction of reduced order mass–damping–stiffness models from identified state space realizations. In particular, it has been shown that

(1) For non–classically damped systems, it is possible to retrieve statically condensed versions of the mass and stiffness matrices. Such a condensation treats the instrumented DOFs (instrumentation in the form of acceleration, velocity, and/or position sensors and force actuators) as the "independent" DOFs, while the remaining DOFs are treated as the "dependent" DOFs and condensed.

(2) If the system has a block diagonal mass matrix, it is possible to exactly construct the partitions of the full order mass and damping matrices corresponding to the instrumented DOFs. Furthermore, it is also possible to obtain an estimate of the partition of the full order stiffness matrix corresponding to the instrumented DOFs, although this estimate is not exact.

(3) Provided that the partition of the mass normalized eigenvectors corresponding to the instrumented DOFs is available, it is possible to exactly identify the partitions of the mass, the damping, and the stiffness matrices corresponding to those instrumented DOFs for classically damped systems with diagonal mass matrices.

The theoretical presentation has been supplemented with a numerical example that illustrates the applicability of the proposed methodology and various issues regarding the formulations developed in this study.

It is anticipated that the methodology presented herein may be applied to various problems in mechanics including finite element model updating and health monitoring of systems with insufficient information. These issues and other investigations such as the effects of noise perturbations on the identified parameters are the subjects of current research and beyond the scope and intent of this study, and hence they will be addressed and reported in future work.

A Proof regarding the diagonalization in the asymmetric eigenvalue problem

Consider the real symmetric generalized eigenvalue problem given by

$$\mathbf{A}\mathbf{\Phi}\mathbf{\Theta} = \mathbf{B}\mathbf{\Phi} \tag{40}$$

where $\mathbf{A}$ and $\mathbf{B}$ are two full rank symmetric matrices, $\mathbf{\Phi}$ is the eigenvector matrix, and $\mathbf{\Theta}$ is the eigenvalue matrix. It is known that, even in the case of repeated roots, there exists a unique set of eigenvectors such that

$$\mathbf{\Phi}^T\mathbf{A}\mathbf{\Phi} = \mathbf{I} \tag{41a}$$

$$\mathbf{\Phi}^T\mathbf{B}\mathbf{\Phi} = \mathbf{\Theta} \tag{41b}$$

and that $\mathbf{\Theta}$ is a diagonal matrix. Now consider a reformulation of this problem as

$$\mathbf{\Psi}\widehat{\mathbf{\Theta}} = \mathbf{A}^{-1}\mathbf{B}\mathbf{\Psi} \tag{42}$$

where $\widehat{\mathbf{\Theta}}$ is the corresponding eigenvalue matrix, and $\mathbf{\Psi}$ is the corresponding eigenvector matrix. Since the product of two symmetric matrices is not necessarily symmetric, eq.(42) is in general an asymmetric eigenvalue problem, and diagonalizability in an asymmetric eigenvalue problem is not guaranteed. Since, however, in this case the asymmetric eigenvalue problem of eq.(42) is in fact derived from the symmetric eigenvalue problem of eqs.(40) and (41), there indeed exist sets of eigenvectors that will yield a diagonal $\widehat{\mathbf{\Theta}}$. In fact, assume that $\mathbf{\Psi} = \mathbf{\Phi}\mathbf{T}$, such that

$$\begin{aligned}
\mathbf{\Psi}^{-1}\mathbf{A}^{-1}\mathbf{B}\mathbf{\Psi} &= (\mathbf{\Phi}\mathbf{T})^{-1}\mathbf{A}^{-1}\mathbf{B}\mathbf{\Phi}\mathbf{T} \\
&= \mathbf{T}^{-1}\mathbf{\Phi}^{-1}\mathbf{\Phi}\mathbf{\Phi}^T\mathbf{B}\mathbf{\Phi}\mathbf{T} && \text{via eq.(41a)} \\
&= \mathbf{T}^{-1}\mathbf{\Theta}\mathbf{T} && \text{via eq.(41b)}
\end{aligned}$$

Therefore, for any diagonal transformation matrix $\mathbf{T}$, the eigenvector matrix $\mathbf{\Psi} = \mathbf{\Phi}\mathbf{T}$ yields $\mathbf{\Psi}^{-1}\mathbf{A}^{-1}\mathbf{B}\mathbf{\Psi} = \widehat{\mathbf{\Theta}} = \mathbf{\Theta}$, and hence the existence of $\mathbf{\Phi}$ implies the diagonalizability of the asymmetric eigenvalue problem in eq.(42).

B Proof that the Moore–Penrose pseudo–inverse of the known partition of the mass normalized eigenvector matrix is equal to a partition of the inverse of the mass normalized eigenvector matrix

The mass normalized eigenvector matrix $\boldsymbol{\phi}_{N\times N}$ can be partitioned such that

$$\boldsymbol{\phi} = \begin{bmatrix} \widehat{\boldsymbol{\phi}} \\ \overline{\boldsymbol{\phi}} \end{bmatrix} = \mathbf{U}\mathbf{S}\mathbf{V}^T = \begin{bmatrix} \widehat{\mathbf{U}}\mathbf{S}\mathbf{V}^T \\ \overline{\mathbf{U}}\mathbf{S}\mathbf{V}^T \end{bmatrix} \tag{43}$$

where $\mathbf{U}_{N\times N}$ and $\mathbf{V}_{N\times N}$ are real unitary matrices containing the left and right singular vectors, and $\mathbf{S}_{N\times N}$ is a diagonal matrix containing the singular values. The Moore–Penrose pseudo–inverse of the partition $\widehat{\boldsymbol{\phi}}$, which is denoted here as $\widehat{\boldsymbol{\phi}}^\dagger$, must satisfy the following four conditions:

$$(1)\ \ \widehat{\boldsymbol{\phi}}\widehat{\boldsymbol{\phi}}^\dagger\widehat{\boldsymbol{\phi}} = \widehat{\boldsymbol{\phi}} \tag{44a}$$

$$(2)\ \ \widehat{\boldsymbol{\phi}}^\dagger\widehat{\boldsymbol{\phi}}\widehat{\boldsymbol{\phi}}^\dagger = \widehat{\boldsymbol{\phi}}^\dagger \tag{44b}$$

$$(3)\ \ \widehat{\boldsymbol{\phi}}\widehat{\boldsymbol{\phi}}^\dagger = (\widehat{\boldsymbol{\phi}}\widehat{\boldsymbol{\phi}}^\dagger)^T \tag{44c}$$

$$(4)\ \ \widehat{\boldsymbol{\phi}}^\dagger\widehat{\boldsymbol{\phi}} = (\widehat{\boldsymbol{\phi}}^\dagger\widehat{\boldsymbol{\phi}})^T \tag{44d}$$

Let us assume that $\widehat{\boldsymbol{\phi}}^\dagger = \mathbf{V}\mathbf{S}^{-1}\widehat{\mathbf{U}}^T$, and check if this solution satisfies the necessary conditions. Keeping in mind that $\mathbf{U}$ and $\mathbf{V}$ are unitary matrices, and that $\widehat{\boldsymbol{\phi}} = \widehat{\mathbf{U}}\mathbf{S}\mathbf{V}^T$, we get

$$(1)\ \ \widehat{\boldsymbol{\phi}}\mathbf{V}\mathbf{S}^{-1}\widehat{\mathbf{U}}^T\widehat{\boldsymbol{\phi}} = \widehat{\mathbf{U}}\mathbf{S}\mathbf{V}^T = \widehat{\boldsymbol{\phi}} \tag{45a}$$

$$(2)\ \ \mathbf{V}\mathbf{S}^{-1}\widehat{\mathbf{U}}^T\widehat{\boldsymbol{\phi}}\mathbf{V}\mathbf{S}^{-1}\widehat{\mathbf{U}}^T = \mathbf{V}\mathbf{S}^{-1}\widehat{\mathbf{U}}^T \tag{45b}$$

$$(3)\ \ \widehat{\boldsymbol{\phi}}\mathbf{V}\mathbf{S}^{-1}\widehat{\mathbf{U}}^T = \widehat{\mathbf{U}}\widehat{\mathbf{U}}^T = (\widehat{\boldsymbol{\phi}}\mathbf{V}\mathbf{S}^{-1})^T \tag{45c}$$

$$(4)\ \ \widehat{\boldsymbol{\phi}}^\dagger\widehat{\boldsymbol{\phi}} = \mathbf{V}\mathbf{S}^{-1}\widehat{\mathbf{U}}^T\widehat{\mathbf{U}}\mathbf{S}\mathbf{V}^T;\, (\widehat{\boldsymbol{\phi}}^\dagger\widehat{\boldsymbol{\phi}})^T = \mathbf{V}\mathbf{S}\widehat{\mathbf{U}}^T\widehat{\mathbf{U}}\mathbf{S}^{-1}\mathbf{V}^T \tag{45d}$$

Note that the proposed solution $\widehat{\boldsymbol{\phi}}^\dagger = \mathbf{V}\mathbf{S}^{-1}\widehat{\mathbf{U}}^T$ satisfies the first three conditions, but it does not satisfy the fourth condition *unless* $\widehat{\mathbf{U}}^T\widehat{\mathbf{U}}$ is a diagonal matrix, and so in general this solution does not hold. In the case of a diagonal mass matrix, however, the fourth condition will also be satisfied. To show the validity of this claim, let us write the expression for the inverse of mass matrix in terms of the normal eigenvectors to yield

$$\boldsymbol{\mathcal{M}}^{-1} = \boldsymbol{\phi}\boldsymbol{\phi}^T = \mathbf{U}\mathbf{S}\mathbf{V}^T\mathbf{V}\mathbf{S}\mathbf{U}^T = \mathbf{U}\mathbf{S}^2\mathbf{U}^T \tag{46}$$

Therefore the columns of $\mathbf{U}$ are the eigenvectors for the eigenvalue problem

$$\boldsymbol{\mathcal{M}}^{-1}\mathbf{U} = \mathbf{U}\mathbf{S}^2 \tag{47}$$

and $\mathbf{U}$ does not have a prescribed structure for a general mass matrix. For a diagonal mass matrix, on the other hand, $\boldsymbol{\mathcal{M}}^{-1}$ is diagonal, and so is $\mathbf{U}$; in fact, in such a case,

$$\mathbf{U}(i, j) = \pm\delta_{ij} \tag{48}$$

where $\mathbf{U}(i, j)$ refers to the element on the i^{th} row and j^{th} column of $\mathbf{U}$, and δ_{ij} is the Kronecker delta. In this case, therefore, the product $\widehat{\mathbf{U}}^T\widehat{\mathbf{U}}$ becomes

$$\widehat{\mathbf{U}}^T\widehat{\mathbf{U}} = \begin{bmatrix} \mathbf{I}_{n\times n} & \mathbf{0}_{n\times(N-n)} \\ \mathbf{0}_{(N-n)\times n} & \mathbf{0}_{(N-n)\times(N-n)} \end{bmatrix} \tag{49}$$

and hence diagonal, such that condition (4) is also satisfied. In conclusion, for a system with a diagonal mass matrix, the proposed solution $\widehat{\boldsymbol{\phi}}^{\dagger} = \mathbf{V}\mathbf{S}^{-1}\widehat{\mathbf{U}}^T$ indeed satisfies all the necessary conditions and hence is the true solution.

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