

# Constructing Second–Order Models of Mechanical Systems From Identified State Space Realizations.

## Part I: Theoretical Discussions.

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### Abstract

This article and its companion study constitute a two part attempt at providing a common framework for some methodologies used in obtaining physical parameters of mechanical systems from identified state space models. In the theoretical part, the authors discuss the basic ideas and formulations involved in such algorithms. It is argued that the problem of extracting the physical matrices of the underlying second order model is equivalent to the identification of certain linear transformation matrices, and that each of the evaluated approaches differs from the others mainly in the particular physically meaningful state space model it chooses as its objective. The choice of this state space model, and the formulations employed therein, define the applicability of the methodologies in terms of the number of sensors and/or actuators they require for a complete identification.

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## 1 Introduction

Modeling and identification of dynamic systems are invariably the two sides of the same coin. Recent advances in algorithms and computing capabilities have resulted in numerous new methodologies in system identification, and many new studies have been reported in various technical journals. This notwithstanding, relations between certain proposed approaches are not always investigated in detail, and different disciplines use different formulations to model similar, and in some cases identical, problems.

One such case is the use of first order differential equations and/or second order differential equations to model the dynamics of mechanical systems. The first order formulation is generally preferred by researchers working in control theory, and various system identification algorithms have been proposed and used in this context (Ewins, 1984; Ibrahim and Mikulcik, 1977; Ibrahim, 1977; Vold et al., 1982; Juang and Pappa, 1985; Juang et al., 1988, 1993; Luş et al., 1999; Luş, 2001; Luş et al., 2002). The result of these methodologies are either the complex modal parameters of the underlying system, or a predictive model in state space, from which the modal parameters can also be obtained. In some other cases, as for example in health monitoring and damage detection, it may be of more interest to consider the physical properties of the underlying mechanical system. Even though some methodologies exist for retrieving such parameters directly from input/output data (see, e.g., Agbabian et al. (1991) and Smyth et al. (2000)), the general practice is to update and calibrate an initial analytical model using the results of dynamic tests (Mottershead and Friswell, 1993; Berman, 1979; Baruch, 1982; Beck and Katafygiotis, 1998). Although these two paradigms are seemingly exclusive, they are related through what has come to be known as the “(linear) inverse vibration problem”.

The so called inverse vibration problem is the determination of the physical parameters, or the coefficient matrices, of second order mechanical models using identified complex model data and/or identified state space realizations. It is well known that, given a set of second order differential equations, one can easily construct a corresponding set of first order equations in quite a straight forward fashion, and this problem is sometimes referred as the “forward problem”. The inverse problem, on the other hand, is much more complex, and has attracted the attention of various researchers as evidenced by numerous publications (Sestieri and Ibrahim, 1994; Imregun and Ewins, 1993; Ibrahim, 1983a; Alvin, 1993; Ibrahim, 1983b; Alvin and Park, 1994; Zhang and Lallement, 1987; Yang and Yeh, 1990; Alvin et al., 1995; Tseng et al., 1994a,b;

Chen et al., 1996; Balmès, 1997; DeAngelis et al., 2002). For a more mathematical treatment of the subject, the reader is referred to the works of Lancaster and Maroulas (1987), and Starek and Inman (1997, 2001).

In this study, we try to provide a unifying framework for some of the approaches that determine the physical parameters of second order dynamic models from identified state space realizations. In particular, we discuss in detail the methodologies proposed by Alvin and Park (1994), Tseng et al. (1994a,b), and DeAngelis et al. (2002). These methodologies, representative of the main body of work in this topic, are chosen because: 1) they all start from a state space realization of the second order mechanical system, 2) they can handle all types of measurements (i.e. displacements, velocities, and/or accelerations), and 3) they are inherently capable of handling the non-proportional viscous damping case.

There are two main issues that need to be addressed in working with such inverse algorithms. The first issue is the one regarding the number of sensors and/or actuators required for a full order solution ('full order' refers to a solution that spans all the degrees of freedom (DOFs) of the underlying mechanical system). In this respect, the algorithm proposed by Alvin and Park (1994) requires a full set of sensors (i.e. all DOFs are measured), while the one proposed by Tseng et al. (1994a,b) requires either a full set of sensors or a full set of actuators. A more general solution is provided by the algorithm proposed by DeAngelis et al. (2002), which requires that all DOFs be instrumented with either a sensor or an actuator.

The second issue is that of having a co-located sensor-actuator pair. In a strict sense, co-location means having both an actuator and a sensor at a certain DOF. For systems that are inherently discrete, this condition may be satisfied in the strict sense of the word; however, the definition of co-location may be slightly modified for continuous systems. If one considers, for example, a finite element discretization of a continuous system, in this case it is possible to apply a force not at a DOF but at an intermediate point. In such a case, the co-location is not identically satisfied; nevertheless, the input matrix, i.e. how this force that is applied at an intermediate point affects various DOFs of the model, is known. Such a general co-location requirement, i.e. knowing the input matrix of the second-order model, is central to all the approaches since it is required either for the identification of the mass matrix via the mass normalized undamped modes, or for the proper scaling of the complex eigenvectors. Even though all the algorithms to be discussed in this work are capable of working with general input matrices, in the course of the discussions we may require

co-location in the strict sense of the word for ease of presentation.

In this theoretical part of the study, we present the key points of the derivations involved in the aforementioned algorithms, and show that these seemingly different approaches can be viewed within a general framework. In particular, it will be argued that 1) all three algorithms differ mainly in the physically meaningful state space coordinates they choose to work with, and 2) the way they choose to articulate the transformations leading to these coordinates determines the requirements they impose on the number of sensors and/or actuators. Issues such as sensitivity to perturbations and algorithmic complexity are to be discussed in detail in the companion paper on numerical discussions (Luş et al., 2002c).

## 2 Definitions and Basic Formulations

### 2.1 Finite Element Representation

The linear, time invariant,  $N$  DOF system considered in this work is one for which the equation of motion and the output equation can be written in matrix form as

$$\mathcal{M}\ddot{\mathbf{q}}(t) + \mathcal{D}\dot{\mathbf{q}}(t) + \mathcal{K}\mathbf{q}(t) = \mathcal{B}\mathbf{u}(t) \quad (1a)$$

$$\mathbf{y}(t) = \begin{bmatrix} \mathcal{C}_p\mathbf{q}(t) \\ \mathcal{C}_v\dot{\mathbf{q}}(t) \\ \mathcal{C}_a\ddot{\mathbf{q}}(t) \end{bmatrix} \quad (1b)$$

where  $\mathcal{M} \in \mathfrak{R}^{N \times N}$ ,  $\mathcal{D} \in \mathfrak{R}^{N \times N}$ , and  $\mathcal{K} \in \mathfrak{R}^{N \times N}$  are the symmetric positive definite mass, damping, and stiffness matrices, respectively, and  $\mathbf{q}(t)$  represents generalized nodal displacements with  $(\dot{\cdot})$  and  $(\ddot{\cdot})$  denoting first and second order derivatives with respect to time. The matrix  $\mathcal{B} \in \mathfrak{R}^{N \times r}$  is the input matrix that relates the excitations in the input vector  $\mathbf{u}(t) \in \mathfrak{R}^{r \times 1}$  to the degrees of freedom (DOFs). The measurement vector  $\mathbf{y}(t) \in \mathfrak{R}^{m \times 1}$  can, in general, include position, velocity, and/or acceleration measurements obtained at various DOFs, and is related to these nodal quantities through the output matrices  $\mathcal{C}_p$ ,  $\mathcal{C}_v$ , and  $\mathcal{C}_a$ , respectively.

The so called *normal modes* for this system are obtained by solving the following generalized eigenvalue

problem:

$$\mathcal{M}\phi_i\Omega_i^2 = \mathcal{K}\phi_i \quad (2)$$

where  $\Omega_i^2$  is the square of the  $i^{th}$  undamped natural frequency, and  $\phi_i$  is the corresponding  $i^{th}$  undamped (normal) eigenvector (mode shape). Although the scaling of the eigenvectors is somewhat arbitrary, arguably the most widely used scaling choice in modal analysis is such that

$$\phi^T \mathcal{M} \phi = \mathbf{I} \quad (3a)$$

$$\phi^T \mathcal{K} \phi = \Omega^2 \quad (3b)$$

in which  $\phi = [\phi_1, \phi_2, \dots, \phi_N] \in \mathfrak{R}^{N \times N}$ ,  $\Omega^2 = \text{diag}(\Omega_1^2, \Omega_2^2, \dots, \Omega_N^2) \in \mathfrak{R}^{N \times N}$ , and  $\mathbf{I}$  is the identity matrix (in this case, with dimensions  $N \times N$ ). With the scaling introduced in eqs.(3), the system in eqs.(1) can be transformed to a set of modal coordinates via a transformation of the form  $\mathbf{q}(t) = \phi\eta(t)$  to yield

$$\ddot{\eta}(t) + \mathcal{E}\dot{\eta}(t) + \Omega^2\eta(t) = \phi^T \mathcal{B} \quad (4a)$$

$$\mathbf{y}(t) = \begin{bmatrix} \mathcal{C}_p \phi \eta(t) \\ \mathcal{C}_v \phi \dot{\eta}(t) \\ \mathcal{C}_a \phi \ddot{\eta}(t) \end{bmatrix} \quad (4b)$$

If the damping matrix in these modal coordinates, which is denoted by  $\mathcal{E}$ , is diagonal, the system is referred to as being *classically damped* (or it is said to have modal damping). In general, the matrix  $\mathcal{E}$  is not diagonal, and the system is then referred to as being a *non-classically damped* system.

On the other hand, the *damped modes* for the system of eqs.(1) are obtained by solving the complex eigenvalue problem:

$$(\mathcal{M}\lambda_i^2 + \mathcal{D}\lambda_i + \mathcal{K})\psi_i = \mathbf{0} \quad (5)$$

An efficient way to approach this problem is to transform it to a symmetric generalized eigenvalue problem, as discussed in detail in the following section.

## 2.2 State Space Representations

By defining a state vector  $\mathbf{x}_1(t)$  as  $\mathbf{x}_1(t) = [\mathbf{q}(t)^T \dot{\mathbf{q}}(t)^T]^T$ , it is possible to transform the system of eqs.(1) to a first order state space representation as

$$\begin{aligned}\dot{\mathbf{x}}_1(t) &= \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathcal{M}^{-1}\mathcal{K} & -\mathcal{M}^{-1}\mathcal{D} \end{bmatrix} \mathbf{x}_1(t) + \begin{bmatrix} \mathbf{0} \\ \mathcal{M}^{-1}\mathcal{B} \end{bmatrix} \mathbf{u}(t) \\ &= \mathbf{A}_1 \mathbf{x}_1(t) + \mathbf{B}_1 \mathbf{u}(t)\end{aligned}\quad (6a)$$

$$\mathbf{y}(t) = [\mathcal{C}_p \quad \mathbf{0}] \mathbf{x}_1(t) = \mathbf{C}_1 \mathbf{x}_1(t) \quad (6b)$$

It should be noted that in the output equation (6b) we have only considered position measurements. This is for ease of exposition only; in subsequent sections the generalization of the following discussions to other types of outputs (namely velocities and accelerations) will be presented. Note also that we have omitted the direct transmission term  $\mathbf{D}\mathbf{u}(t)$  in eq.(6b), simply because it is invariant under coordinate transformations and does not get involved in the algorithms to be discussed in this study.

Analogously, if we define a state vector using the modal displacements and modal velocities, i.e. by defining  $\mathbf{x}_2(t) = [\boldsymbol{\eta}(t)^T \dot{\boldsymbol{\eta}}(t)^T]^T$ , eqs.(4) can be transformed to

$$\begin{aligned}\dot{\mathbf{x}}_2(t) &= \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\boldsymbol{\Omega}^2 & -\boldsymbol{\mathcal{E}} \end{bmatrix} \mathbf{x}_2(t) + \begin{bmatrix} \mathbf{0} \\ \boldsymbol{\phi}^T \mathcal{B} \end{bmatrix} \mathbf{u}(t) \\ &= \mathbf{A}_2 \mathbf{x}_2(t) + \mathbf{B}_2 \mathbf{u}(t)\end{aligned}\quad (7a)$$

$$\mathbf{y}(t) = [\mathcal{C}_p \boldsymbol{\phi} \quad \mathbf{0}] \mathbf{x}_2(t) = \mathbf{C}_2 \mathbf{x}_2(t) \quad (7b)$$

Following the terminology employed by Alvin and Park (1994), we will call the representation in eqs.(6) the *displacement - velocity model* (DV), and similarly, the representation in eqs.(7) the *modal displacement - velocity model* (MDV). For both of these systems the eigenvalue problems are non-symmetric with the same set of eigenvalues. To differentiate between the two formulations, the eigenvalue problem for the DV model will be denoted by

$$\begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathcal{M}^{-1}\mathcal{K} & -\mathcal{M}^{-1}\mathcal{D} \end{bmatrix} \begin{bmatrix} \boldsymbol{\psi}^d \\ \boldsymbol{\psi}^d \boldsymbol{\Lambda} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\psi}^d \\ \boldsymbol{\psi}^d \boldsymbol{\Lambda} \end{bmatrix} \boldsymbol{\Lambda} \quad (8)$$

where the matrix  $\boldsymbol{\psi}^d = [\boldsymbol{\psi}_1^d \boldsymbol{\psi}_2^d \dots \boldsymbol{\psi}_{2N}^d]_{N \times 2N}$  contains the complex valued eigenvectors  $\boldsymbol{\psi}_i^d$ , and the diagonal matrix  $\boldsymbol{\Lambda}_{2N \times 2N} = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_{2N})$  contains the complex valued eigenvalues of the following

eigenvalue problem:

$$(\mathbf{I}\lambda_i^2 + \mathcal{M}^{-1}\mathcal{D}\lambda_i + \mathcal{M}^{-1}\mathcal{K})\psi_i^d = 0 \quad (9)$$

with  $i = 1, \dots, 2N$ . By considering a transformation of the form  $\mathbf{x}_1(t) = [(\psi^d)^T \quad (\psi^d \mathbf{\Lambda})^T]^T \hat{\mathbf{x}}_1(t)$ , eqs.(6) can be re-written as

$$\dot{\hat{\mathbf{x}}}_1(t) = \mathbf{\Lambda} \hat{\mathbf{x}}_1(t) + \begin{bmatrix} \psi^d \\ \psi^d \mathbf{\Lambda} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{0} \\ \mathcal{M}^{-1} \mathcal{B} \end{bmatrix} \mathbf{u}(t) \quad (10a)$$

$$\mathbf{y}(t) = \mathcal{C}_p \psi^d \hat{\mathbf{x}}_1(t) \quad (10b)$$

Similarly, the eigenvalue problem for the MDV model will be denoted by

$$\begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{\Omega}^2 & -\mathcal{E} \end{bmatrix} \begin{bmatrix} \psi^m \\ \psi^m \mathbf{\Lambda} \end{bmatrix} = \begin{bmatrix} \psi^m \\ \psi^m \mathbf{\Lambda} \end{bmatrix} \mathbf{\Lambda} \quad (11)$$

where the matrix  $\psi^m = [\psi_1^m \ \psi_2^m \ \dots \ \psi_{2N}^m]_{N \times 2N}$  contains the complex valued eigenvectors  $\psi_i^m$  for the MDV representation, which can also be obtained via

$$(\mathbf{I}\lambda_i^2 + \mathcal{E}\lambda_i + \mathbf{\Omega}^2)\psi_i^m = 0 \quad (12)$$

while the eigenvalues in  $\mathbf{\Lambda}$  are identical to the eigenvalues in eq.(8) since both eq.(8) and eq.(11) belong to the same second order system. Also note that the scaling of the eigenvectors in both eqs.(8) and eq.(11) can be somewhat arbitrary. The scaling factors show up, in an inverse fashion, only in the input and output matrices of the first order models. Here we assume, without loss of generality, that the scaling is chosen so as to make the Euclidean norm of each eigenvector equal to 1.

Yet another formulation becomes apparent when one considers a state space representation using a state vector  $\mathbf{x}_3(t) = [\mathbf{q}(t)^T \dot{\mathbf{q}}(t)^T]^T$  such that

$$\begin{bmatrix} \mathcal{D} & \mathcal{M} \\ \mathcal{M} & \mathbf{0} \end{bmatrix} \dot{\mathbf{x}}_3(t) + \begin{bmatrix} \mathcal{K} & \mathbf{0} \\ \mathbf{0} & -\mathcal{M} \end{bmatrix} \mathbf{x}_3(t) = \begin{bmatrix} \mathcal{B} \\ \mathbf{0} \end{bmatrix} \mathbf{u}(t) \quad (13a)$$

$$\mathbf{y}(t) = [\mathcal{C}_p \quad \mathbf{0}] \mathbf{x}_3(t) \quad (13b)$$

The advantage of this formulation, compared to the model in eqs.(6), is that the related generalized eigenvalue problem is symmetric, i.e.

$$\begin{bmatrix} \mathcal{D} & \mathcal{M} \\ \mathcal{M} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \psi \\ \psi \Lambda \end{bmatrix} \Lambda = \begin{bmatrix} -\mathcal{K} & \mathbf{0} \\ \mathbf{0} & \mathcal{M} \end{bmatrix} \begin{bmatrix} \psi \\ \psi \Lambda \end{bmatrix} \quad (14)$$

where the matrix  $\psi = [\psi_1 \ \psi_2 \ \dots \ \psi_{2N}]_{N \times 2N}$  contains the eigenvectors of the symmetric generalized eigenvalue problem of eq.(14). Once again, the eigenvalues in  $\Lambda$  in eq.(14) are identical to those presented in eqs.(8) and (11). If the eigenvectors in  $\psi$  are scaled such that (see Sestieri and Ibrahim (1994), Balmès (1997))

$$\begin{bmatrix} \psi \\ \psi \Lambda \end{bmatrix}^T \begin{bmatrix} \mathcal{D} & \mathcal{M} \\ \mathcal{M} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \psi \\ \psi \Lambda \end{bmatrix} = \mathbf{I} \quad (15a)$$

$$\begin{bmatrix} \psi \\ \psi \Lambda \end{bmatrix}^T \begin{bmatrix} -\mathcal{K} & \mathbf{0} \\ \mathbf{0} & \mathcal{M} \end{bmatrix} \begin{bmatrix} \psi \\ \psi \Lambda \end{bmatrix} = \Lambda \quad (15b)$$

then, by considering the transformation  $\mathbf{x}_3(t) = [\psi^T \ (\psi \Lambda)^T]^T \hat{\mathbf{x}}_3(t)$ , eqs.(13) can be re-written as

$$\dot{\hat{\mathbf{x}}}_3(t) = \Lambda \hat{\mathbf{x}}_3(t) + \psi^T \mathcal{B} \mathbf{u}(t) \quad (16a)$$

$$\mathbf{y}(t) = \mathcal{C}_p \psi \hat{\mathbf{x}}_3(t) \quad (16b)$$

To differentiate this representation from the previous ones, we will refer to it as the *symmetric formulation*.

The main advantage of this formulation is that, instead of the inverse of the eigenvector matrix, as is the case with  $([(\psi^d)^T \ (\psi^d \Lambda)^T]_{2N \times 2N}^T)^{-1}$  in eq.(10a), the transpose of the eigenvector matrix  $\psi_{N \times 2N}$  shows up in eq.(16a). This seemingly simple modification will have important consequences in terms of the number of instruments (i.e. number of sensors and actuators) required for the final solution.

All these formulations are just different representations of the same mechanical system, namely the second order model of eqs.(1). In the forthcoming sections, it will be discussed in detail that the various algorithms reviewed in this work differ in the way they choose to formulate the problem, i.e. in the particular state space models they choose to work with. The inherent common framework in all these algorithms is the fact that they essentially aim to find some transformation matrices that will allow them to start from an arbitrary representation and change the coordinates to one of the above mentioned sets.

One last note we should mention is on the type of dynamic problem considered in this study. In order to keep the discussion compact and concise, for both this work and the companion paper on numerical investigations, we assume that the first order eigenvalues are distinct and appear in complex conjugate pairs. This statement is equivalent to saying that all the modes of the underlying second order system are underdamped, and that there are no rigid body modes. It is also assumed that, even in the asymmetric cases, the related state matrices are diagonalizable. This requirement is satisfied if there are no repeated roots (first order eigenvalues), but it may also be extended to the case of repeated roots. If all the first order eigenvalues  $\lambda_i$  ( $i = 1$  to  $2N$ ) in eqs.(8), (11), and (14) appear in complex conjugate pairs, i.e. if

$$\lambda_{2i} = \lambda_{2i-1}^*; \quad i = 1, 2, \dots, N$$

where the superscript (\*) denotes the complex conjugate, then the related eigenvectors have the same property:

$$\psi_{2i}^d = (\psi_{2i-1}^d)^*; \quad \psi_{2i}^m = (\psi_{2i-1}^m)^*; \quad \psi_{2i} = (\psi_{2i-1})^*; \quad i = 1, 2, \dots, N$$

Furthermore, we will assume that all  $N$  modes of the second order model are identified during dynamic testing, so that the number of modes will be equal to the number of DOFs.

### 2.3 Identified State Space Model

For the purposes of this study, we assume that a state space model of the dynamic system is identified from general input - output data using an ERA/OKID based approach (for the details of such a technique, the reader is referred to the works of Juang et al. (1988, 1993) and Luş et al. (1999, 2002)). If the dynamic test was performed with  $r$  inputs and  $m$  outputs, and if all  $N$  modes of the underlying second order model could be identified, the result of such an identification is a discrete time state space realization that can be expressed as

$$\mathbf{z}(k+1) = \mathbf{\Phi}\mathbf{z}(k) + \mathbf{\Gamma}\mathbf{u}(k) \quad (17a)$$

$$\mathbf{y}(k) = \mathbf{C}\mathbf{z}(k) + \mathbf{D}\mathbf{u}(k) \quad (17b)$$

where  $k$  is the time index,  $\mathbf{\Phi}_{2N \times 2N}$  is the state transition matrix,  $\mathbf{\Gamma}_{2N \times r}$  is the discrete time input matrix, and  $\mathbf{C}_{m \times 2N}$  is the discrete time output matrix. Once again we choose to omit the direct transmission term

$\mathbf{D}\mathbf{u}(k)$  in the following presentation due to reasons previously discussed.

The continuous time representation of this discrete time system can be formulated as

$$\dot{\mathbf{z}}(t) = \mathbf{A}_C \mathbf{z}(t) + \mathbf{B}_C \mathbf{u}(t) \quad (18a)$$

$$\mathbf{y}(t) = \mathbf{C}_C \mathbf{z}(t) \quad (18b)$$

The matrix  $\mathbf{A}_C$ , of dimensions  $2N \times 2N$ , is related to its discrete time counterpart via the well known matrix exponential  $\Phi = e^{\mathbf{A}_C \Delta T}$ , where  $\Delta T$  is the sampling time interval. The relation between  $\mathbf{B}_C$  and  $\Gamma$  is slightly more involved, since here one must employ an assumption as to how the data is sampled (e.g. zero order hold), while the output matrix  $\mathbf{C}_C$  is identical to its discrete time counterpart  $\mathbf{C}$ .

It is also possible to express these equations in some modal coordinates. If the matrix  $\mathbf{A}_C$  is diagonalized by a matrix  $\varphi$  such that  $\varphi^{-1} \mathbf{A}_C \varphi = \Lambda$ , then a transformation of the form  $\mathbf{z}(t) = \varphi \hat{\mathbf{z}}(t)$  leads to

$$\dot{\hat{\mathbf{z}}}(t) = \Lambda \hat{\mathbf{z}}(t) + \varphi^{-1} \mathbf{B}_C \mathbf{u}(t) \quad (19a)$$

$$\mathbf{y}(t) = \mathbf{C}_C \varphi \hat{\mathbf{z}}(t) \quad (19b)$$

The starting point for the following discussions is that if the data used in the identification really came from the second order system of eqs.(1), then the model of eqs.(19) and the models discussed in Section 2.2 are just different representations of the same system. Hence we are now ready to discuss some of the most prominent algorithms that aim to identify the physical parameters (namely the mass, damping, and stiffness matrices) of the second order model starting from the arbitrary state space representation of eqs.(18), or equivalently, eqs.(19).

### 3 Algorithm I: The Case of Full Set of Sensors

The first algorithm to be discussed was proposed initially by Alvin and Park (1994), and the presentation here is analogous to the one by Bernal and Gunes (2002). This algorithm provides a solution when the number of sensors is equal to the number of identified second order modes (or, for the purposes of this study, to the DOFs). Since all the DOFs are instrumented with a sensor, there will be (at least) one actuator co-located with a sensor.

Here we discuss a slightly different formulation of the approach which can be more easily incorporated into the general framework of this study. The key point in the formulations is the comparison of the models of eqs.(6) and (19). Since these two models are just different representations of the same system, there must be a transformation matrix  $\mathbb{T}_1$ , of dimensions  $2N \times 2N$ , such that

$$\mathbf{x}_1(t) = \mathbb{T}_1 \widehat{\mathbf{z}}(t) \quad (20)$$

which leads to

$$\mathbb{T}_1 \mathbf{A} = \mathbf{A}_1 \mathbb{T}_1; \quad \mathbb{T}_1^{-1} \mathbf{B}_1 = \boldsymbol{\varphi}^{-1} \mathbf{B}_C; \quad \mathbf{C}_1 \mathbb{T}_1 = \mathbf{C}_C \boldsymbol{\varphi} \quad (21)$$

By investigating the transformation for the matrix  $\mathbf{A}_1$ , we can conclude that the transformation matrix we are looking for must consist of the eigenvectors  $\boldsymbol{\psi}^d$ , the eigenvalues  $\mathbf{\Lambda}$ , and perhaps also a diagonal matrix  $\widetilde{\mathbb{T}}_1$  that contains some unknown scaling factors for the eigenvectors, i.e.

$$\mathbb{T}_1 = \begin{bmatrix} \boldsymbol{\psi}^d \\ \boldsymbol{\psi}^d \mathbf{\Lambda} \end{bmatrix} \widetilde{\mathbb{T}}_1 = \begin{bmatrix} \boldsymbol{\psi}^d \widetilde{\mathbb{T}}_1 \\ \boldsymbol{\psi}^d \mathbf{\Lambda} \widetilde{\mathbb{T}}_1 \end{bmatrix} \quad (22)$$

The solution is developed by considering the transformation for the output matrix in eq.(21). If there exists a full set of (independent) sensors, then the output matrix  $\mathbf{C}_p$  in  $\mathbf{C}_1$  is square and full rank (rank  $N$ ), and so the product  $\boldsymbol{\psi}^d \widetilde{\mathbb{T}}_1$  can be evaluated from

$$\mathbf{C}_1 \mathbb{T}_1 = \mathbf{C}_p \boldsymbol{\psi}^d \widetilde{\mathbb{T}}_1 = \mathbf{C}_C \boldsymbol{\varphi} \Rightarrow \boldsymbol{\psi}^d \widetilde{\mathbb{T}}_1 = \mathbf{C}_p^{-1} \mathbf{C}_C \boldsymbol{\varphi} \quad (23)$$

Furthermore, since both  $\mathbf{\Lambda}$  and  $\widetilde{\mathbb{T}}_1$  are diagonal,  $\boldsymbol{\psi}^d \widetilde{\mathbb{T}}_1 \mathbf{\Lambda} = \boldsymbol{\psi}^d \mathbf{\Lambda} \widetilde{\mathbb{T}}_1$ , which implies that

$$\begin{bmatrix} \boldsymbol{\psi}^d \\ \boldsymbol{\psi}^d \mathbf{\Lambda} \end{bmatrix} \widetilde{\mathbb{T}}_1 = \begin{bmatrix} \mathbf{C}_p^{-1} \mathbf{C}_C \boldsymbol{\varphi} \\ \mathbf{C}_p^{-1} \mathbf{C}_C \boldsymbol{\varphi} \mathbf{\Lambda} \end{bmatrix} \quad (24)$$

where all the entries on the right hand side of eq.(24) are known. Now the state matrix  $\mathbf{A}_1$  can be evaluated using the expression in eq.(8), i.e.

$$\begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathcal{M}^{-1} \mathcal{K} & -\mathcal{M}^{-1} \mathcal{D} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\psi}^d \\ \boldsymbol{\psi}^d \mathbf{\Lambda} \end{bmatrix} \mathbf{\Lambda} \begin{bmatrix} \boldsymbol{\psi}^d \\ \boldsymbol{\psi}^d \mathbf{\Lambda} \end{bmatrix}^{-1} = \begin{bmatrix} \boldsymbol{\psi}^d \\ \boldsymbol{\psi}^d \mathbf{\Lambda} \end{bmatrix} \widetilde{\mathbb{T}}_1 \mathbf{\Lambda} \widetilde{\mathbb{T}}_1^{-1} \begin{bmatrix} \boldsymbol{\psi}^d \\ \boldsymbol{\psi}^d \mathbf{\Lambda} \end{bmatrix}^{-1} \quad (25)$$

in which the last equality holds since  $\widetilde{\mathbb{T}}_1$  is diagonal.

When solving the inverse problem, it is noteworthy that the partition  $\tilde{\mathbb{T}}_1$  does not need to be explicitly identified to construct the state space matrices of the DV model. Furthermore, once the state matrix is evaluated from eq.(25), it is possible to obtain the undamped eigenvalues and the undamped eigenvectors of the underlying second order model by solving

$$\mathcal{M}^{-1}\mathcal{K}\widehat{\phi}_i = \Omega_i^2\widehat{\phi}_i; \quad i = 1, 2, \dots, N \quad (26)$$

where  $\widehat{\phi}_i$  represents the *arbitrarily scaled*  $i^{th}$  normal eigenvector, and  $\mathcal{M}^{-1}\mathcal{K}$  is a partition of the now known  $\mathbf{A}_1$ . Now the co-location requirement becomes essential because, in order to be able to extract the mass matrix  $\mathcal{M}$  from the DV model, one needs to know the mass normalized eigenvectors  $\phi_i$  for  $i = 1, 2, \dots, N$ . The relation between the arbitrarily scaled eigenvectors  $\widehat{\phi}_i$  and the mass normalized eigenvectors  $\phi_i$  can be obtained by exploiting the co-location requirement.

To elucidate the foregoing discussion, let us assume that the two sets of eigenvectors are related through a scaling matrix  $\mathcal{S}$  such that  $\phi = \widehat{\phi}\mathcal{S}$ , where the previously defined matrix  $\phi$  contains the mass normalized undamped eigenvectors, the matrix  $\widehat{\phi} = [\widehat{\phi}_1 \widehat{\phi}_2 \dots \widehat{\phi}_N]_{N \times N}$  contains the arbitrarily scaled undamped eigenvectors, and the diagonal matrix  $\mathcal{S} = \text{diag}(s_1, s_2, \dots, s_N)$  contains the scaling factors. Using the scaling condition of eq.(3a), the inverse of the mass matrix can be evaluated via  $\mathcal{M}^{-1} = (\phi\phi^T)$  which can be rewritten using the arbitrarily scaled eigenvectors and the unknown scaling factors as

$$\mathcal{M}^{-1} = \widehat{\phi}\mathcal{S}^2\widehat{\phi}^T = s_1^2\widehat{\phi}_1\widehat{\phi}_1^T + s_2^2\widehat{\phi}_2\widehat{\phi}_2^T + \dots + s_N^2\widehat{\phi}_N\widehat{\phi}_N^T \quad (27)$$

From the transformation equations in eq.(21) we have  $\mathbb{T}_1^{-1}\mathbf{B}_1 = \varphi^{-1}\mathbf{B}_C$ , or equivalently,

$$\begin{bmatrix} \psi^d \\ \psi^d\Lambda \end{bmatrix} \tilde{\mathbb{T}}_1\varphi^{-1}\mathbf{B}_C = \begin{bmatrix} \mathcal{C}_p^{-1}\mathbf{C}_C\varphi \\ \mathcal{C}_p^{-1}\mathbf{C}_C\varphi\Lambda \end{bmatrix} \varphi^{-1}\mathbf{B}_C = \begin{bmatrix} \mathbf{0} \\ \mathcal{M}^{-1}\mathcal{B} \end{bmatrix} \quad (28)$$

The lower partition of this equation leads to

$$\begin{aligned} \mathcal{C}_p^{-1}\mathbf{C}_C\varphi\Lambda\varphi^{-1}\mathbf{B}_C &= s_1^2\widehat{\phi}_1\widehat{\phi}_1^T\mathcal{B} + s_2^2\widehat{\phi}_2\widehat{\phi}_2^T\mathcal{B} + \dots + s_N^2\widehat{\phi}_N\widehat{\phi}_N^T\mathcal{B} \\ &= \left[ \widehat{\phi}_1\widehat{\phi}_1^T\mathcal{B}; \widehat{\phi}_2\widehat{\phi}_2^T\mathcal{B}; \dots; \widehat{\phi}_N\widehat{\phi}_N^T\mathcal{B} \right] \begin{bmatrix} s_1^2 \\ s_2^2 \\ \vdots \\ s_N^2 \end{bmatrix} \end{aligned}$$

such that, assuming that there is only one input, the scaling factors can be obtained from

$$[s_1^2 \ s_2^2 \ \dots \ s_N^2]^T = [\hat{\phi}_1 \hat{\phi}_1^T \mathbf{B}; \hat{\phi}_2 \hat{\phi}_2^T \mathbf{B}; \dots; \hat{\phi}_N \hat{\phi}_N^T \mathbf{B}]^{-1} \mathbf{C}_p^{-1} \mathbf{C}_C \varphi \Lambda \varphi^{-1} \mathbf{B}_C \quad (29)$$

If there are  $r > 1$  inputs then the matrices  $\mathbf{B}$  and  $\mathbf{B}_C$  have  $r$  columns. Since the preceding discussion is valid independently for each of the inputs (columns), eq.(29) needs to be solved for each column, leading to  $r$  sets of scaling factors. If there is no noise in the system, all the scaling factors come out to be the same, i.e. only one set suffices and the rest are redundant. On the other hand, in the presence of noise, such sets will not be identical, and it would be best to proceed with a least squares solution for the scaling factors by employing the information from each of the columns simultaneously. Also note that the requirement on the input matrix is not that of a strict co-location, but rather that of a general input matrix which is known a priori.

Once the scaling factors are obtained from eq.(29), the mass normalized eigenvectors are evaluated from  $\phi = \hat{\phi} \mathcal{S}$ , and the mass matrix is obtained via  $\mathcal{M} = (\phi \phi^T)^{-1}$ . Using the mass matrix, the damping and stiffness matrices can be constructed by evaluating

$$\mathcal{D} = -\mathcal{M}(-\mathcal{M}^{-1}\mathcal{D}); \quad \mathcal{K} = -\mathcal{M}(-\mathcal{M}^{-1}\mathcal{K}) \quad (30)$$

where  $(-\mathcal{M}^{-1}\mathcal{D})$  and  $(-\mathcal{M}^{-1}\mathcal{K})$  are known partitions from eq.(25).

A last note on this algorithm is about the changes in the methodology when there are different types of measurements. If one uses velocity or acceleration measurements, the difference is in the output equation of the DV model of eqs.(6), i.e.

$$\mathbf{y}(t) = [\mathbf{0} \ \mathbf{C}_v] \mathbf{x}_1(t) \quad (31a)$$

$$\mathbf{y}(t) = [\mathbf{0} \ \mathbf{C}_a] \dot{\mathbf{x}}_1(t) = [\mathbf{0} \ \mathbf{C}_a] \mathbf{A}_1 \mathbf{x}_1(t) (+\mathbf{D}_C \mathbf{u}(t)) \quad (31b)$$

Using the eigenvalue decomposition of  $\mathbf{A}_1$  and the transformation relations, it is possible to show that eq.(23) becomes

$$\psi^{d\tilde{\mathbb{T}}_1} = \mathbf{C}_v^{-1} \mathbf{C}_C \varphi \Lambda^{-1} \quad \text{for velocity measurements} \quad (32a)$$

$$\psi^{d\tilde{\mathbb{T}}_1} = \mathbf{C}_a^{-1} \mathbf{C}_C \varphi \Lambda^{-2} \quad \text{for acceleration measurements} \quad (32b)$$

When there is a full set of sensors, the required partition  $\psi^{d\tilde{\mathbb{T}}_1}$  is evaluated from either eq.(32a) or eq.(32b), depending on the type of measurement, and the algorithm proceeds in exactly the same way as for the displacement measurements. More generally, the output vector can contain mixed types of sensors, and each row of the eigenvector matrix must be evaluated separately by incorporating the necessary modifications.

#### 4 Algorithm 2: The Case of Full Set of Sensors or Full Set of Actuators

A more general methodology was presented in a series of papers by Tseng et al. (1994a,b), in which solutions for the cases with either a full set of actuators or a full set of sensors were presented. Here we will discuss the basic principles of this approach, and try to highlight its inherent differences from the other algorithms.

The starting points for this algorithm are: 1) the transformation of the model in eqs.(19) to the McMillan normal form, and 2) the rearranging of the equations, so that the model in eqs.(19) can be expressed in a new set of coordinates as

$$\begin{aligned}\dot{\mathbf{z}}_M(t) &= \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{\Omega}_M^2 & -\mathbf{\Sigma} \end{bmatrix} \mathbf{z}_M(t) + \begin{bmatrix} \mathbf{B}_{1M} \\ \mathbf{B}_{2M} \end{bmatrix} \mathbf{u}(t) \\ &= \mathbf{A}_{CM} \mathbf{z}_M(t) + \mathbf{B}_{CM} \mathbf{u}(t)\end{aligned}\quad (33a)$$

$$\mathbf{y}(t) = [\mathbf{C}_{1M} \quad \mathbf{C}_{2M}] \mathbf{z}_M(t) = \mathbf{C}_{CM} \mathbf{z}_M(t) \quad (33b)$$

If we denote by  $\sigma_i$  the real part and by  $\omega_i$  the imaginary part of the  $i^{th}$  first order eigenvalue  $\lambda_i$ , then the matrix  $\mathbf{\Omega}_M^2 \in \mathfrak{R}^{N \times N}$  is diagonal with elements of the type  $\sigma_i^2 + \omega_i^2$ , and the matrix  $\mathbf{\Sigma} \in \mathfrak{R}^{N \times N}$  is also a diagonal matrix whose elements are  $-2\sigma_i$ , for  $i = 1, 2, \dots, N$  (assuming that the eigenvalues appear in complex conjugate pairs as discussed before). The McMillan normal form is important for modal analysis because, for *classically damped* systems, the matrices  $\mathbf{\Omega}_M^2$  and  $\mathbf{\Sigma}$  contain the normal modal parameters, i.e.

$$\begin{aligned}\Omega_i &= \sqrt{\sigma_i^2 + \omega_i^2} \\ \zeta_i &= \frac{-\sigma_i}{\Omega_i}\end{aligned} \Leftrightarrow \sigma_i \pm j\omega_i = -\zeta_i \Omega_i \pm j\Omega_i \sqrt{1 - \zeta_i^2}; \quad i = 1, 2, \dots, N \quad (34)$$

where  $\Omega_i$  is the  $i^{th}$  undamped natural frequency,  $\zeta_i$  is the corresponding modal damping percentage, and  $j$  denotes the imaginary number  $\sqrt{-1}$ . When the system is not classically damped, the modal parameters calculated using eq.(34) are generally meaningless. If we now compare the state space representation of the

second order modal model in eq.(7) with the model in eqs.(33), the following differences can be observed for a general system:

- $\Omega_M^2 = \Omega^2$  only if the system is undamped or classically damped
- $\mathcal{E}$  is symmetric, but does not have to be diagonal as  $\Sigma$
- The input and output matrices  $\mathbf{B}_2$  and  $\mathbf{C}_2$  must contain the zero partitions as shown in eq.(7), but these partitions are not necessarily zero in matrices  $\mathbf{B}_{CM}$  and  $\mathbf{C}_{CM}$  in eq.(33)

However, since they do represent the same system, there must be a transformation matrix  $\mathbb{T}_2$ , of dimensions  $2N \times 2N$ , that relates these two representations through  $\mathbf{x}_2(t) = \mathbb{T}_2 \mathbf{z}_M(t)$ . The transformation equations can now be stated as

$$\mathbb{T}_2 \mathbf{A}_{CM} = \mathbf{A}_2 \mathbb{T}_2; \quad \mathbb{T}_2^{-1} \mathbf{B}_2 = \mathbf{B}_{CM}; \quad \mathbf{C}_2 \mathbb{T}_2 = \mathbf{C}_{CM} \quad (35)$$

#### 4.1 Finding the Transformation Matrix $\mathbb{T}_2$

The procedure to obtain the transformation matrix  $\mathbb{T}_2$  is quite cumbersome in terms of algebraic manipulations. Here we present the key points of the derivations and provide some original discussions in interpreting the results. For the mathematical details omitted in this presentation, the reader is referred to the works by Tseng et al. (1994a,b). Let us start by considering the following partitioning of the matrix  $\mathbb{T}_2$  such that

$$\mathbb{T}_2 = \mathbf{N}\mathbf{R} = \begin{bmatrix} \mathbf{N}_{11} & \mathbf{N}_{12} \\ \mathbf{N}_{21} & \mathbf{N}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{R}_{11} & \mathbf{R}_{12} \\ \mathbf{R}_{21} & \mathbf{R}_{22} \end{bmatrix} \quad (36)$$

where  $\mathbf{N}_{ij}$  and  $\mathbf{R}_{ij}$  ( $i$  and  $j = 1, 2$ ) are the  $N \times N$  partitions of the matrices  $\mathbf{N}$  and  $\mathbf{R}$ , respectively. Using these partitions, eqs.(35) can be rewritten as

$$\mathbf{N}\mathbf{R} \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\Omega_M^2 & -\Sigma \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\Omega^2 & -\mathcal{E} \end{bmatrix} \mathbf{N}\mathbf{R} \quad (37a)$$

$$\mathbf{N}\mathbf{R} \begin{bmatrix} \mathbf{B}_{1M} \\ \mathbf{B}_{2M} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \phi^T \mathcal{B} \end{bmatrix} \quad (37b)$$

$$[\mathbf{C}_{1M} \quad \mathbf{C}_{2M}] = [\mathbf{C}_p \phi \quad \mathbf{0}] \mathbf{N} \mathbf{R} \quad (37c)$$

The first step in the solution is to impose the zero partitions in the input and output matrices. Let us first assume that the number of independent actuators is equal to the number of second order modes (i.e.  $r = N$ ).

In this case

$$\text{rank}(\mathbf{B}_{CM}) = \text{rank} \left( \begin{bmatrix} \mathbf{B}_{1M} \\ \mathbf{B}_{2M} \end{bmatrix} \right) = N$$

and the singular value decomposition of  $\mathbf{B}_{CM}$  can be represented as

$$\begin{bmatrix} \mathbf{B}_{1M} \\ \mathbf{B}_{2M} \end{bmatrix} = \mathbf{U}_b \begin{bmatrix} \mathbf{S}_b \\ \mathbf{0} \end{bmatrix} \mathbf{V}_b^T = \mathbf{U}_b \begin{bmatrix} \mathbf{S}_b \mathbf{V}_b^T \\ \mathbf{0} \end{bmatrix} = \mathbf{U}_b \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ \mathbf{S}_b \mathbf{V}_b^T \end{bmatrix} \quad (38)$$

Since the number of unknown matrix partitions in eqs.(37) is greater than the number of available matrix equations, it is convenient to select the matrix  $\mathbf{R}$  as

$$\mathbf{R}^T = \mathbf{U}_b \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{bmatrix} \quad (39)$$

By definition of the singular value decomposition,  $\mathbf{R}$  now has the property that  $\mathbf{R}^T = \mathbf{R}^{-1}$ . With this particular choice of  $\mathbf{R}$ , eq.(37b) leads to

$$\mathbf{N}_{12} \mathbf{S}_b \mathbf{V}_b^T = \mathbf{0} \quad (40)$$

$$\mathbf{N}_{22} \mathbf{S}_b \mathbf{V}_b^T = \phi^T \mathcal{B} \quad (41)$$

which implies that  $\mathbf{N}_{12} = \mathbf{0}$ . Furthermore, in order to have  $\text{rank}(\mathbb{T}_2) = 2N$ , now the partitions  $\mathbf{N}_{11}$  and  $\mathbf{N}_{22}$  must both be full rank ( $\text{rank}(\mathbf{N}_{11}) = \text{rank}(\mathbf{N}_{22}) = N$ ). With this choice of  $\mathbf{R}$ , the transformation equation for the output matrix leads to

$$\mathbf{C}_{1M} \mathbf{R}_{11}^T + \mathbf{C}_{2M} \mathbf{R}_{12}^T = \mathbf{C}_p \phi \mathbf{N}_{11} \quad (42)$$

$$\mathbf{C}_{1M} \mathbf{R}_{21}^T + \mathbf{C}_{2M} \mathbf{R}_{22}^T = \mathbf{0} \quad (43)$$

By inspecting eqs. (40)-(43), we can conclude that if the data comes from a mechanical system with force or torque inputs and displacement outputs, eq. (43) must be satisfied automatically, i.e. there is no extra

freedom to explicitly impose this condition. However, it is very important to note that eq. (43) will not be identically satisfied when noise is present in the measurements.

If, on the other hand, one has  $N$  position sensors instead of  $N$  actuators, the singular value decomposition is performed on the output matrix as opposed to the input matrix, such that

$$[\mathbf{C}_{1M} \quad \mathbf{C}_{2M}] = \mathbf{U}_C [\mathbf{S}_C \quad \mathbf{0}] \mathbf{V}_C^T = [\mathbf{U}_C \mathbf{S}_C \quad \mathbf{0}] \mathbf{V}_C^T$$

In this case, the matrix  $\mathbf{R}$  is chosen as  $\mathbf{R} = \mathbf{V}_C^T$ , and once again it has the property that  $\mathbf{R}^{-1} = \mathbf{R}^T$ . Also in this case  $\mathbf{N}_{12} = \mathbf{0}$ , and the product  $\mathbf{R}\mathbf{B}_{CM}$  leads to

$$\phi^T \mathbf{B} = \mathbf{N}_{11} (\mathbf{N}_{11}^{-1} \mathbf{N}_{22}) (\mathbf{R}_{21} \mathbf{B}_{1M} + \mathbf{R}_{22} \mathbf{B}_{2M})$$

In either case, once the initial partition  $\mathbf{R}$  is chosen accordingly, it is possible to evaluate some partitions of the matrix  $\mathbf{N}$ . In fact, by rewriting  $\mathbf{N}$  as

$$\mathbf{N} = \begin{bmatrix} \mathbf{N}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{N}_{11} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{N}_{11}^{-1} \mathbf{N}_{21} & \mathbf{N}_{11}^{-1} \mathbf{N}_{22} \end{bmatrix}$$

it can be shown, using some algebra, that eq.(37a) can be rewritten as

$$\begin{bmatrix} \mathbf{N}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{N}_{11} \end{bmatrix} \begin{bmatrix} \mathbf{W}_{11} & \mathbf{W}_{12} \\ \mathbf{W}_{21} & \mathbf{W}_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\Omega^2 & -\mathcal{E} \end{bmatrix} \begin{bmatrix} \mathbf{N}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{N}_{11} \end{bmatrix} \quad (44)$$

where  $\mathbf{W}_{ij}$  ( $i$  and  $j = 1, 2$ ) are the  $N \times N$  partitions of a new matrix  $\mathbf{W}$  that contains all known quantities. The top partition of eq.(44) does not give us any new information ( $\mathbf{W}_{11} = \mathbf{0}$  and  $\mathbf{W}_{12} = \mathbf{I}$ ), but the bottom partition leads to the following matrix equations:

$$\mathbf{N}_{11} \mathbf{W}_{21} = -\Omega^2 \mathbf{N}_{11} \quad (45)$$

$$\mathbf{N}_{11} \mathbf{W}_{22} = -\mathcal{E} \mathbf{N}_{11} \quad (46)$$

Equation (45) represents an eigenvalue problem: the eigenvalues in the diagonal matrix  $\Omega^2$  are the squares of the undamped natural frequencies of the second order model, while the rows of the eigenvector matrix  $\mathbf{N}_{11}$  are the left eigenvectors. Unfortunately the solution to such an eigenvalue problem is not unique in the following senses:

- Each eigenvector can be multiplied by an arbitrary non-zero complex number,
- The order of the eigenvalues in  $-\Omega^2$  can be changed by changing the order of the eigenvectors in  $\mathbf{N}_{11}$ .

To get a unique answer, two additional conditions need to be imposed, namely that

- The “modal damping” matrix  $\mathcal{E}$  does not have to be diagonal, but it has to be symmetric,
- The input and output matrices must be real.

These issues will be addressed in detail in section 4.2. For sake of completeness, let us now consider the case where, instead of position measurements, one uses velocity or acceleration measurements. Since the state vector in eq.(7) contains the modal displacements and velocities ( $\mathbf{x}_2(t) = [\boldsymbol{\eta}^T(t) \quad \dot{\boldsymbol{\eta}}^T(t)]^T$ ), it can be easily shown (simply by substitution) that

$$\begin{aligned} [\mathbf{0} \quad \mathbf{C}_v \boldsymbol{\phi}] \mathbf{x}_2(t) &= [\mathbf{C}_v \boldsymbol{\phi} \quad \mathbf{0}] \mathbf{A}_2 \mathbf{x}_2(t) \\ [\mathbf{0} \quad \mathbf{C}_a \boldsymbol{\phi}] \dot{\mathbf{x}}_2(t) &= [\mathbf{C}_a \boldsymbol{\phi} \quad \mathbf{0}] \mathbf{A}_2^2 \mathbf{x}_2(t) (+\mathbf{D}_C \mathbf{u}(t)) \end{aligned}$$

Keeping in mind that  $\mathbb{T}_2 \mathbf{A}_{CM} = \mathbf{A}_2 \mathbb{T}_2$ , the transformation equations for the output matrix can be written as

$$\begin{aligned} \mathbf{C}_{CM} &= \mathbf{C}_2 \mathbb{T}_2 = [\mathbf{C}_v \boldsymbol{\phi} \quad \mathbf{0}] \mathbf{A}_2 \mathbb{T}_2 = [\mathbf{C}_v \boldsymbol{\phi} \quad \mathbf{0}] \mathbb{T}_2 \mathbf{A}_{CM} \rightarrow \text{for velocities} \\ \mathbf{C}_{CM} &= \mathbf{C}_2 \mathbb{T}_2 = [\mathbf{C}_a \boldsymbol{\phi} \quad \mathbf{0}] \mathbf{A}_2^2 \mathbb{T}_2 = [\mathbf{C}_a \boldsymbol{\phi} \quad \mathbf{0}] \mathbb{T}_2 \mathbf{A}_{CM}^2 \rightarrow \text{for accelerations} \end{aligned}$$

so that the singular value decomposition can be performed on modified matrices, i.e.

$$\begin{aligned} \mathbf{C}_{CM} \mathbf{A}_{CM}^{-1} &= [\mathbf{U}_C \mathbf{S}_C \quad \mathbf{0}] \mathbf{V}_C^T \rightarrow \text{for velocities} \\ \mathbf{C}_{CM} \mathbf{A}_{CM}^{-2} &= [\mathbf{U}_C \mathbf{S}_C \quad \mathbf{0}] \mathbf{V}_C^T \rightarrow \text{for accelerations} \end{aligned}$$

With these modifications, the new representations are called “position equivalent representations”, and the solution proceeds in exactly the same way as for the case of position sensors.

## 4.2 The Symmetry of the Modal Damping Matrix

The conditions for the unique solution of the eigenvectors of eq.(45), namely that the modal damping matrix must be symmetric and that the input and output matrices must be real, are imposed in two consecutive

steps. By imposing one or the other as the first step leads to two slightly modified formulations (more detailed discussions can be found in the works of Tseng et al. (1994a,b)). However, the input and output matrices come out real automatically because the undamped eigenvalues and eigenvectors for a mechanical system are real. Therefore, in the following derivations, we will assume that these two matrices are real (or that the condition of real input-output matrices has been imposed) and discuss the symmetry of the damping matrix.

Let us denote by  $\widehat{\mathbf{N}}_{11}$  an initial set of left eigenvectors for the eigenvalue problem of eq.(45). The rows of this eigenvector matrix are the arbitrarily scaled eigenvectors  $\widehat{\mathbf{n}}_i^T$  ( $i = 1, 2, \dots, N$ ). The unknown matrix  $\mathbf{N}_{11}$  is proportional to the matrix  $\widehat{\mathbf{N}}_{11}$  in the sense that the rows of the matrix  $\mathbf{N}_{11}$  can be written as  $\alpha_i \widehat{\mathbf{n}}_i^T$ , where the  $\alpha_i$  is the scaling factor for the  $i^{th}$  left eigenvector  $\widehat{\mathbf{n}}_i^T$ . Since the condition for real input-output matrices has already been imposed, the  $\alpha_i$ 's are real quantities, and now  $\mathbf{N}_{11}$  can be written as  $\mathbf{N}_{11} = \text{diag}(\boldsymbol{\alpha})\widehat{\mathbf{N}}_{11}$  where  $\text{diag}(\boldsymbol{\alpha}) = \text{diag}(\alpha_1, \alpha_2, \dots, \alpha_N)$  is diagonal matrix.

To impose the symmetry of  $\mathcal{E}$ , the property that  $\mathcal{E} - \mathcal{E}^T = \mathbf{0}$  will be imposed, so that pre-multiplying eq.(46) by  $\mathbf{N}_{11}^T$  and post-multiplying the transpose of eq.(46) by  $\mathbf{N}_{11}$  will lead to

$$\mathbf{N}_{11}^T \mathbf{N}_{11} \mathbf{W}_{22} - \mathbf{W}_{22}^T \mathbf{N}_{11}^T \mathbf{N}_{11} = \mathbf{0} \quad (47)$$

Eq.(47) contains  $N^2$  scalar equations, but instead of tackling these equations directly, it is possible to rearrange them in a more convenient form. In fact, it was shown by Tseng et al. (1994a,b) that the symmetry condition can be re-written as

$$\mathbf{T}_{N^2 \times N} \begin{bmatrix} \alpha_1^2 \\ \vdots \\ \alpha_N^2 \end{bmatrix}_{N \times 1} = \mathbf{0} \quad (48)$$

where  $\mathbf{T}$  is a matrix whose rank depends on the number of modes that are coupled, either directly or indirectly, with each other. Such a matrix must have its rank less than  $N$  because of eq.(47). Its rank can at most be  $N - 1$ , corresponding to the case where all the modes of the second order model are coupled. On the other hand, if the second order model is classically damped, its rank will be 0, meaning that any choice of scaling factors will preserve the symmetry of the damping matrix. It is very important to note that the symmetry condition allows us to determine only the *ratios of the scaling factors, and not their absolute*

magnitudes. This issue is addressed in the next section.

The appropriate ratios of the scaling factors can now be obtained using the null space vectors of the matrix  $\mathbf{T}$ . Although this is a well defined approach, there arise some problems when one works with a noisy system: in fact, it may well happen that the null space does not even exist when enough noise is present in the measurements. Once the ratios of the scaling factors are determined, the unknown partition of the transformation matrix, the modal damping matrix  $\mathcal{E}$ , and the input and output matrices in some modal coordinates, can be constructed. Note that since the magnitudes of the scaling factors are as yet undetermined, all these matrices are constructed with respect to some global scaling factors, and this issue will be investigated in the next section.

### 4.3 Uniqueness of the Modal Model

As mentioned in the previous section, the symmetry of the modal damping matrix  $\mathcal{E}$  is imposed by choosing appropriate ratios for the scaling factors  $\alpha_i$ . However, the magnitudes of these scaling factors are still not uniquely determined, since different magnitudes can lead to same ratios and still preserve the symmetry of  $\mathcal{E}$ . In this sense, the identified modal model is not unique.

Consider now that the scaling factors in  $diag(\alpha)$  can be written as the product of two matrices as  $diag(\alpha) = \mathcal{S}diag(\hat{\alpha})$ , where  $\mathcal{S} = diag(s_1, s_2, \dots, s_N)$  contains the undetermined magnitudes, and  $diag(\hat{\alpha})$  contains the ratios that lead to a symmetric damping matrix in modal coordinates. Let us now consider a modal model of the second order system, obtained by the method described in the previous sections and pick the matrix  $\mathcal{S}$  as the identity matrix. Such a modal model can be represented by

$$\ddot{\eta}_1(t) + \mathcal{E}\dot{\eta}_1(t) + \Omega^2\eta_1(t) = \hat{\mathcal{B}}\mathbf{u}(t) \quad (49a)$$

$$\mathbf{y}(t) = \hat{\mathcal{C}}_p\eta_1(t) \quad (49b)$$

where  $\hat{\mathcal{B}}$  and  $\hat{\mathcal{C}}_p$  are the input and output matrices in these modal coordinates (note that they are not equal to  $\phi^T\mathcal{B}$  and  $\mathcal{C}_p\phi$ , respectively, since the scaling factors, and hence the transformation matrix, have not been truly determined yet). On the other hand, using the definition that  $diag(\alpha) = \mathcal{S}diag(\hat{\alpha})$ , “another” modal

model of the same second order system can be written as

$$\ddot{\eta}_2(t) + \mathcal{S}^{-1} \mathcal{E} \mathcal{S} \dot{\eta}_2(t) + \Omega^2 \eta_2(t) = \mathcal{S}^{-1} \widehat{\mathcal{B}} \mathbf{u}(t) \quad (50a)$$

$$\mathbf{y}(t) = \widehat{\mathcal{C}}_p \mathcal{S} \eta_2(t) \quad (50b)$$

If the system is classically damped, i.e. if the modal damping matrix  $\mathcal{E}$  is diagonal, each eigenvector can be scaled independently, and all these scaling factors would preserve the symmetry condition. In this case, each  $s_i$  would be different, but the symmetric viscous damping terms in equations (49) and (50) (namely  $\mathcal{E}$  and  $\mathcal{S}^{-1} \mathcal{E} \mathcal{S}$ ) would be equal. On the other hand, if all the modes are coupled, then there is only one independent scaling factor  $s_i$  that can be chosen to preserve the symmetry of the damping matrix in the modal model; but also in this case, the symmetric viscous damping terms  $\mathcal{E}$  and  $\mathcal{S}^{-1} \mathcal{E} \mathcal{S}$  would be equal, since every coefficient in the matrix  $\mathcal{E}$  would essentially be multiplied by 1. In more general terms, if a set of modes are coupled and each mode in this set is uncoupled with all the modes that are excluded from this set, there will be a single scaling factor  $s_i$  corresponding to this set. It was discussed by Tseng et al. (1994a,b) that for a noise free system, the number of sets (or groups) is directly related to the rank of the matrix  $\mathbf{T}$  in eq.(48). In any case, the symmetric viscous damping term is identical for both eqs.(49) and (50), so that the general modal model can be written as

$$\ddot{\eta}(t) + \mathcal{E} \dot{\eta}(t) + \Omega^2 \eta(t) = \mathcal{S}^{-1} \widehat{\mathcal{B}} \mathbf{u}(t) \quad (51a)$$

$$\mathbf{y}(t) = \widehat{\mathcal{C}}_p \mathcal{S} \eta(t) \quad (51b)$$

and the undetermined scaling factors in  $\mathcal{S}$  appear only in the input and output matrices.

#### 4.4 Identification of the mass, damping, and stiffness matrices

What are left to be identified in terms of modal quantities are the mass normalized eigenvectors  $\phi$  and the scaling factors  $\mathcal{S}$  from eq.(51). In order to identify these unknowns, let us start by investigating the modal representations in eqs.(4) and (51); clearly, the differences between these two representations are in the input and output matrices, i.e.

$$\mathcal{S}^{-1} \widehat{\mathcal{B}} = \phi^T \mathcal{B}; \quad \widehat{\mathcal{C}}_p \mathcal{S} = \mathcal{C}_p \phi \quad (52)$$

The matrices  $\widehat{\mathbf{B}}$  and  $\widehat{\mathbf{C}}_p$  are known from the results of previous discussions. Analogous to Algorithm 1 discussed in section 3, we assume that the matrices  $\mathbf{B}$  and  $\mathbf{C}_p$  are known, and that the rank of at least one of these matrices is equal to the number of second order modes (which is an assumption we previously employed for obtaining the modal model of eq.(51)). This statement is equivalent to saying that there are either a full set of sensors ( $rank(\mathbf{C}_p) = N$ ) or a full set of actuators ( $rank(\mathbf{B}) = N$ ). Using eqs.(52), the procedure to obtain the mass normalized eigenvectors  $\phi$  and the unknown scaling factors  $\mathcal{S}$  can be presented as follows:

- When the input matrix  $\mathbf{B}$  is full rank (the case of  $N$  actuators), then

$$\phi = (\widehat{\mathbf{B}}\mathbf{B}^{-1})^T \mathcal{S}^{-1} \quad (53)$$

$$\widehat{\mathbf{C}}_p = \mathbf{C}_p (\widehat{\mathbf{B}}\mathbf{B}^{-1})^T \mathcal{S}^{-2} \quad (54)$$

In this case, the scaling factors are solved from eq.(54) and are then substituted in eq.(53) for the evaluation of the mass normalized eigenvectors  $\phi$ .

- When the output matrix  $\mathbf{C}_p$  is of full rank (the case of  $N$  sensors), then

$$\phi = (\mathbf{C}_p^{-1} \widehat{\mathbf{C}}_p) \mathcal{S} \quad (55)$$

$$\widehat{\mathbf{B}} = \mathcal{S}^2 (\mathbf{C}_p^{-1} \widehat{\mathbf{C}}_p)^T \mathbf{B} \quad (56)$$

In this case, the scaling factors are solved from eq.(56) and are then substituted in eq.(55) for the evaluation of the mass normalized eigenvectors  $\phi$ .

If there is a single output for  $N$  actuators or a single actuator for  $N$  outputs (i.e. one co-located sensor-actuator pair), the scaling factors are solved uniquely from eqs.(54) or (56). However, in the presence of noise with more sensors or actuators, it is best to find a least squares solution to these factors so as to minimize the effects of noise.

Once the mass normalized eigenvectors are evaluated, the mass, damping, and stiffness matrices of the FE model can be evaluated using

$$\mathcal{M} = (\phi\phi^T)^{-1}; \quad \mathcal{D} = \phi^{-T} \mathcal{E} \phi^{-1}; \quad \mathcal{K} = \phi^{-T} \Omega^2 \phi^{-1} \quad (57)$$

and this completes the solution. Here we should state that one can pick either the positive or the negative root of  $s_i^2$  when these scaling factors are determined from either eq.(54) or (56). However, a sign change in the  $i^{th}$  scaling factor  $s_i$  is equivalent to multiplying the  $i^{th}$  mass normalized eigenvector  $\phi_i$  by a factor of  $\pm 1$ , and therefore has no effect on the final matrices  $\mathcal{M}$ ,  $\mathcal{D}$ , and  $\mathcal{K}$ . An interesting note on this algorithm is the observation that the physical parameters and the transformation matrix  $\mathbb{T}_2$  are identified simultaneously, i.e. the construction of the matrix  $\mathbb{T}_2$  requires the scaling factors  $s_i$ , which in turn must be evaluated simultaneously with the undamped modeshapes  $\phi_i$ . It is also noteworthy that, analogous to Algorithm 1, the formulations require only that the input matrix be known, and that the co-location in the strict sense of the word is not enforced.

## 5 Algorithm 3: Utilizing Mixed Types of Information

Only recently, a procedure which does not require a full set of actuators or a full set of sensors has been proposed by DeAngelis et al. (2002). The requirement for this algorithm is that all DOFs should have either a sensor or an actuator, with at least one DOF containing a co-located sensor-actuator pair. For ease of presentation, the co-location requirement is enforced in the strict sense; however, with slight modifications, the following discussion can be generalized to incorporate the case of a general input matrix. Note that this approach represents a generalization compared to the previous approaches where either a full set of sensors and/or a full set of actuators were required.

Here we denote with  $\mathbf{M}(k, :)$  and with  $\mathbf{M}(:, l)$  the  $k^{th}$  row and the  $l^{th}$  column, respectively, of a generic matrix  $\mathbf{M}$ ; analogously,  $\mathbf{M}(k, l)$  refers to the element on the  $k^{th}$  row and on the  $l^{th}$  column of the matrix  $\mathbf{M}$ . Furthermore, for purposes of book keeping, let us use an artifice that has been successfully adopted in the works of Luş (2001) and DeAngelis et al. (2002). This artifice consists of 1) expanding the input and output matrices for the models in eqs.(13) and (18) to include all the degrees of freedom, and 2) reordering these matrices so that there is a common correspondence with the number of the degrees of freedom. Such an expansion can be easily done by adding, in the proper order, columns of zeros in the input matrices ( $\mathbf{B}_C$  and  $\mathbf{B}$ ) corresponding to the DOFs that do not contain an actuator, and rows of zeros in the output matrices ( $\mathbf{C}_C$  and  $\mathbf{C}_p$ ) for the DOFs at which the response is not measured. With the help of these modifications, the

$i^{th}$  column of the input matrices and the  $i^{th}$  row of the output matrices correspond to the  $i^{th}$  DOF.

The algorithm essentially tries to determine a transformation matrix  $\mathbb{T}_3$  that relates the model in eqs.(19) to the model of eqs.(16) such that  $\hat{\mathbf{x}}_3(t) = \mathbb{T}_3 \hat{\mathbf{z}}(t)$ , and the transformation equations can be expressed as

$$\mathbb{T}_3 \mathbf{\Lambda} = \mathbf{\Lambda} \mathbb{T}_3; \quad \mathbb{T}_3^{-1} \boldsymbol{\psi}^T \mathbf{B} = \boldsymbol{\varphi}^{-1} \mathbf{B}_C; \quad \mathbf{C}_p \boldsymbol{\psi} \mathbb{T}_3 = \mathbf{C}_C \boldsymbol{\varphi} \quad (58)$$

These conditions can also be written in terms of the above mentioned ‘‘expanded’’ matrices as

$$\mathbb{T}_3 \mathbf{\Lambda} = \mathbf{\Lambda} \mathbb{T}_3; \quad \mathbb{T}_3 \boldsymbol{\varphi}^{-1} \mathbf{B}_C^E = \boldsymbol{\psi}^T \mathbf{B}^E; \quad \mathbf{C}_C^E \boldsymbol{\varphi} \mathbb{T}_3^{-1} = \mathbf{C}_p^E \boldsymbol{\psi} \quad (59)$$

where  $\mathbf{B}_C^E$ ,  $\mathbf{B}^E$ ,  $\mathbf{C}_C^E$ , and  $\mathbf{C}_p^E$  denote the expanded versions of the input and output matrices of eq.(58).

With the assumptions (discussed in section 2) employed in this study, it is easy to show that the transformation matrix is diagonal (i.e.  $\mathbb{T}_3 = \text{diag}(t_1, t_2, \dots, t_{2N})$ ) and that its elements are complex conjugate. By examining eqs.(58) and/or (59), it is clear that the matrix  $\mathbb{T}_3$  has a two-fold effect: 1) to transform the eigenvectors from those of a non-symmetric eigenvalue problem ( $\boldsymbol{\varphi}$ ) to those of a symmetric eigenvalue problem ( $\boldsymbol{\psi}$ ), and 2) to properly scale such eigenvectors. In articulating the following discussions, without loss of generality and for ease of presentation, we assume that the input and output matrices ( $\mathbf{B}$  and  $\mathbf{C}_p$ , respectively) of the finite element model contain information only about the geometry of the structure and the locations of the sensors and the actuators, i.e. we assume that they contain binary information with regards to the existence of an actuator and/or a sensor at a particular DOF.

## 5.1 Evaluating the transformation matrix and the properly scaled complex eigenvectors

The identification of the transformation matrix  $\mathbb{T}_3$  and the properly scaled complex eigenvectors  $\boldsymbol{\psi}$  can be investigated by studying a general case, since it can be shown that the case of full set of sensors and the case of full set of actuators are only special cases. Let us assume that each degree of freedom contains *either* an actuator *or* a sensor, with one degree of freedom containing a co-located sensor - actuator pair (hence  $r + m = N + 1$ ). If there is a co-located sensor - actuator pair on the generic  $i^{th}$  DOF, then the co-location requirement can be simply stated as

$$\mathbf{C}_p^E(i, :) \boldsymbol{\psi} = (\boldsymbol{\psi}^T \mathbf{B}^E(:, i))^T \quad (60)$$

Note that in general, this co-location requirement does not need to be satisfied literally, and that the relation between the inputs and outputs can be articulated via properly combining the entries in the related input-output matrices. Now the transformation matrix  $\mathbb{T}_3$  can be evaluated using eqs.(59) and (60) as:

$$\mathbf{C}_C^E(i, :)\boldsymbol{\varphi}\mathbb{T}_3^{-1} = (\mathbb{T}_3\boldsymbol{\varphi}^{-1}\mathbf{B}_C^E(:, i))^T; \Rightarrow \mathbf{C}_C^E(i, :)\boldsymbol{\varphi} = (\boldsymbol{\varphi}^{-1}\mathbf{B}_C^E(:, i))^T\mathbb{T}_3^2 \quad (61)$$

and, since the matrix  $\mathbb{T}_3$  is diagonal, each  $t_i$  ( $i = 1, 2, \dots, 2N$ ) can be uniquely determined from eq.(61).

Once these scaling factors have been evaluated, the complex eigenvector matrix  $\boldsymbol{\psi}$  can be obtained using the expressions in eq.(59), since the information pertaining to a certain degree of freedom is embedded either in the input matrix or in the output matrix. Going back to equations (59), the output matrices essentially contain information about only  $m$  DOFs (with  $m < N$ ). If there is a sensor at the  $k^{th}$  DOF, and since  $\mathbf{C}_p^E(k, :)\boldsymbol{\psi} = \boldsymbol{\psi}(k, :)$ , the  $k^{th}$  row of the matrix  $\boldsymbol{\psi}$  can be evaluated using

$$\boldsymbol{\psi}(k, :) = \mathbf{C}_C^E(k, :)\boldsymbol{\varphi}\mathbb{T}_3^{-1} \quad (62)$$

On the other hand, if there is no sensor at the  $k^{th}$  DOF, then  $\mathbf{C}_p^E(k, :)\boldsymbol{\psi} = \mathbf{0}_{1 \times 2N}$ . However, since a DOF is assumed to be instrumented by either a sensor or an actuator, the  $k^{th}$  row of the matrix  $\boldsymbol{\psi}$  can be evaluated for the case of an actuator using the second identity in eq.(59):

$$\boldsymbol{\psi}(k, :) = (\mathbb{T}_3\boldsymbol{\varphi}^{-1}\mathbf{B}_C^E(:, k))^T \quad (63)$$

Clearly, this argument is valid for all the  $N$  DOFs and so all the rows of the matrix  $\boldsymbol{\psi}$  can be uniquely evaluated. It should be noted that, for the  $i^{th}$  DOF that contains the co-located sensor - actuator pair, one can use either eq.(62) or eq.(63), since they lead to the same result by the co-location requirement in eq.(60).

If there is a full set of sensors ( $rank(\mathbf{C}_p) = N$ ,  $\mathbf{C}_p^E \equiv \mathbf{C}_p$ , and  $\mathbf{C}_C^E \equiv \mathbf{C}_C$ ), or a full set of actuators ( $rank(\mathbf{B}) = N$ ,  $\mathbf{B}^E \equiv \mathbf{B}$ , and  $\mathbf{B}_C^E \equiv \mathbf{B}_C$ ), the scaling factors are still evaluated from eq.(60). Once the scaling factors are evaluated, one can identify the complex eigenvector matrix  $\boldsymbol{\psi}$  using

$$\mathbf{C}_p^{-1}\mathbf{C}_C\boldsymbol{\varphi}\mathbb{T}_3^{-1} = \boldsymbol{\psi} \quad (64)$$

when there is a full set of sensors, or

$$\mathbb{T}_3\boldsymbol{\varphi}^{-1}\mathbf{B}_C\mathbf{B}^{-1} = \boldsymbol{\psi}^T \quad (65)$$

when there is a full set of actuators. Clearly, these two cases can be regarded as special cases of the general formulation presented in this section.

The changes that must be incorporated if one uses velocity or acceleration measurements are analogous to the changes presented for the previous discussions. If velocity measurements are used, eq.(16b) is to be replaced with

$$\mathbf{y}(t) = \begin{bmatrix} \mathbf{0} & \mathbf{C}_v \end{bmatrix} \begin{bmatrix} \boldsymbol{\psi} \\ \boldsymbol{\psi}\boldsymbol{\Lambda} \end{bmatrix} \hat{\mathbf{x}}_3(t) = \mathbf{C}_v \boldsymbol{\psi} \boldsymbol{\Lambda} \hat{\mathbf{x}}_3(t) \quad (66)$$

while for acceleration measurements,

$$\begin{aligned} \mathbf{y}(t) &= \begin{bmatrix} \mathbf{0} & \mathbf{C}_a \end{bmatrix} \begin{bmatrix} \boldsymbol{\psi} \\ \boldsymbol{\psi}\boldsymbol{\Lambda} \end{bmatrix} \dot{\hat{\mathbf{x}}}_3(t) = \begin{bmatrix} \mathbf{0} & \mathbf{C}_a \end{bmatrix} \begin{bmatrix} \boldsymbol{\psi} \\ \boldsymbol{\psi}\boldsymbol{\Lambda} \end{bmatrix} \boldsymbol{\Lambda} \hat{\mathbf{x}}_3(t) + \begin{bmatrix} \mathbf{0} & \mathbf{C}_a \end{bmatrix} \begin{bmatrix} \boldsymbol{\psi} \\ \boldsymbol{\psi}\boldsymbol{\Lambda} \end{bmatrix} \boldsymbol{\psi}^T \mathbf{B} \mathbf{u}(t) \\ &= \mathbf{C}_a \boldsymbol{\psi} \boldsymbol{\Lambda}^2 \hat{\mathbf{x}}_3(t) (+\mathbf{D}_C \mathbf{u}(t)) \end{aligned} \quad (67)$$

These changes leads to some alterations in eq.(58), i.e.

$$\mathbf{C}_C \boldsymbol{\varphi} = \mathbf{C}_v \boldsymbol{\psi} \boldsymbol{\Lambda} \mathbb{T}_3 \quad \text{for velocity measurements} \quad (68a)$$

$$\mathbf{C}_C \boldsymbol{\varphi} = \mathbf{C}_a \boldsymbol{\psi} \boldsymbol{\Lambda}^2 \mathbb{T}_3 \quad \text{for acceleration measurements} \quad (68b)$$

and since the complex eigenvalue matrix  $\boldsymbol{\Lambda}$  is known and both  $\boldsymbol{\Lambda}$  and  $\mathbb{T}_3$  are diagonal matrices, eqs.(68) can be rewritten as

$$\mathbf{C}_C \boldsymbol{\varphi} \boldsymbol{\Lambda}^{-1} \mathbb{T}_3^{-1} = \mathbf{C}_v \boldsymbol{\psi} \quad \text{for velocity measurements} \quad (69a)$$

$$\mathbf{C}_C \boldsymbol{\varphi} \boldsymbol{\Lambda}^{-2} \mathbb{T}_3^{-1} = \mathbf{C}_a \boldsymbol{\psi} \quad \text{for acceleration measurements} \quad (69b)$$

The output matrices  $\mathbf{C}_v$  and  $\mathbf{C}_a$  are analogous to the output matrix  $\mathbf{C}_p$ ; therefore, all we have to do to use the algorithm and discussions presented in this section is to use  $\mathbf{C}_C \boldsymbol{\varphi} \boldsymbol{\Lambda}^{-1}$  instead of  $\mathbf{C}_C \boldsymbol{\varphi}$  in eq.(58) if velocity measurements are used instead of position measurements, or to use  $\mathbf{C}_C \boldsymbol{\varphi} \boldsymbol{\Lambda}^{-2}$  instead of  $\mathbf{C}_C \boldsymbol{\varphi}$  in eq.(58) for the case of acceleration measurements.

## 5.2 Identification of the mass, damping, and stiffness matrices

Once the properly scaled eigenvector matrix  $\boldsymbol{\psi}$  is evaluated, the mass, damping, and stiffness matrices of the finite element model can be obtained using the orthogonality conditions in eqs.(15). It has been previously

discussed (see, e.g., Balmès (1997)) that algebraic manipulations on eqs.(15) lead to the following identities:

$$\begin{bmatrix} \mathcal{D} & \mathcal{M} \\ \mathcal{M} & \mathbf{0} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{0} & \mathcal{M}^{-1} \\ \mathcal{M}^{-1} & -\mathcal{M}^{-1}\mathcal{D}\mathcal{M}^{-1} \end{bmatrix} = \begin{bmatrix} \psi \\ \psi\Lambda \end{bmatrix} \begin{bmatrix} \psi \\ \psi\Lambda \end{bmatrix}^T = \begin{bmatrix} \psi\psi^T & \psi\Lambda\psi^T \\ \psi\Lambda\psi^T & \psi\Lambda^2\psi^T \end{bmatrix} \quad (70a)$$

$$\begin{bmatrix} -\mathcal{K} & \mathbf{0} \\ \mathbf{0} & \mathcal{M} \end{bmatrix}^{-1} = \begin{bmatrix} -\mathcal{K}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathcal{M}^{-1} \end{bmatrix} = \begin{bmatrix} \psi \\ \psi\Lambda \end{bmatrix} \Lambda^{-1} \begin{bmatrix} \psi \\ \psi\Lambda \end{bmatrix}^T = \begin{bmatrix} \psi\Lambda^{-1}\psi^T & \psi\psi^T \\ \psi\psi^T & \psi\Lambda\psi^T \end{bmatrix} \quad (70b)$$

In order for eqs.(70) to be valid, it is necessary that:

$$\mathcal{M} = (\psi\Lambda\psi^T)^{-1}, \quad \mathcal{D} = -\mathcal{M}\psi\Lambda^2\psi^T\mathcal{M}, \quad (71a)$$

$$\mathcal{K} = -(\psi\Lambda^{-1}\psi^T)^{-1}, \quad \psi\psi^T = \mathbf{0} \quad (71b)$$

which provide the required expressions for the mass, damping and stiffness matrices of the second order model of the system. It should be noted, however, that the condition of having  $\psi\psi^T = \mathbf{0}$  will not be identically satisfied in the presence of noise. For further details and discussions, the reader is referred to the works of Luş (2001) and DeAngelis et al. (2002).

## 6 Discussions

In this theoretical section of the two-part study, the authors have attempted to provide a general framework to the problem of extraction of physical parameters of mechanical systems from identified state space representations. Various algorithms have been discussed and key points in their developments have been examined in detail. With regards to the general framework and to the differences in the aforementioned algorithms, the following points should be further commented on:

- (i) The problem under investigation has been articulated as one of constructing some linear transformation matrices that allow one to transform the state space equations in some arbitrary coordinates to physically meaningful coordinate bases. It has been shown that the algorithms reviewed in this study differ in only the final physically meaningful state space coordinates they choose to work with. For Algorithm 1 and Algorithm 3, the coordinate system is the physical displacement-velocity basis, and

for Algorithm 2, the coordinate system is the modal displacement–velocity basis. The difference between Algorithm 1 and Algorithm 3 is that Algorithm 1 works with the asymmetric formulation of eqs.(6), while Algorithm 3 employs the symmetric formulation of eqs.(13)

- (ii) The particular coordinate basis an algorithm employs, together with the requirements such a choice imposes, determines the required number of sensors and/or actuators needed for a complete identification. In this context, it has been discussed that Algorithm 1 requires a full set of sensors, while Algorithm 2 requires either a full set of sensors or a full set of actuators. Instead, Algorithm 3 can handle the more general case of mixed number of sensors and actuators provided that all DOFs are instrumented with either a sensor or an actuator. As previously mentioned, all these algorithms require a co-located sensor–actuator pair. For Algorithm 1 and Algorithm 2, the co-location is used to obtain the mass normalized undamped second order modes, whereas for Algorithm 3, the co-location is required to find the scaling factors for the complex mode shapes.
- (iii) It should be emphasized that the approaches discussed herein are formulated in the time domain and provide, starting from an arbitrary state space formulation, the mass, damping, and stiffness matrices of the underlying second order mechanical system. In addition, such approaches could also be used to obtain the transfer function of the entire structural system. In fact, by taking the Laplace transform of eqs. (16) and by combining the two transformed equations, it is possible to obtain an expression that relates the input transform vector,  $\mathbf{U}(s)$ , and the output transform vector,  $\mathbf{Y}(s)$ , of the  $N$  DOF system as:

$$\mathbf{Y}(s) = \mathbf{C}_p \boldsymbol{\psi} [s\mathbf{I} - \boldsymbol{\Lambda}]^{-1} \boldsymbol{\psi}^T \mathbf{B} \mathbf{U}(s) \quad (72)$$

and analogously, by using (10), we get

$$\mathbf{Y}(s) = \mathbf{C}_p \boldsymbol{\psi}^d [s\mathbf{I} - \boldsymbol{\Lambda}]^{-1} \begin{bmatrix} \boldsymbol{\psi}^d \\ \boldsymbol{\psi}^d \boldsymbol{\Lambda} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{0} \\ \mathcal{M}^{-1} \mathbf{B} \end{bmatrix} \mathbf{U}(s) \quad (73)$$

For ease of reference, both expressions could be written in a more compact and general form as

$$\mathbf{Y}(s) = \mathbf{C}_p \mathbf{H}(s) \mathbf{B} \mathbf{U}(s) \quad (74)$$

where the matrix  $\mathbf{H}(s)$ , of dimension  $N \times N$ , represents the transfer function matrix of the entire  $N$  DOF system. The complete knowledge of  $\mathbf{H}(s)$  would allow one to determine the response of the system at any DOF for an arbitrary input applied at any DOF, creating a complete predictive model of the system. When a dynamic test is performed, unless all DOFs are instrumented with both a sensor and an actuator, only some elements of the transfer function matrix are directly identified. The purpose of the algorithms discussed in this study is essentially to identify the physical second order matrices, but these identified matrices in turn can be used to construct the whole transfer function matrix  $\mathbf{H}(s)$ . For the case of full set of sensors, only a single column of  $\mathbf{H}(s)$  is directly identified, but all three algorithms may be used to construct the whole  $\mathbf{H}(s)$ . For the case of full set of actuators, only a single row of  $\mathbf{H}(s)$  is directly identified, and in this case only Algorithm 2 and Algorithm 3 can provide the solution for the whole transfer function matrix. When there is neither a full set of sensors nor a full set of actuators, but all the DOFs are instrumented with either one or the other, only Algorithm 3 can provide the means for the construction of the whole  $\mathbf{H}(s)$ . The strength of Algorithm 3 lies in the “input–output equivalence” concept (see Luş (2001) and DeAngelis et al. (2002)), implying that for this methodology, there is no distinction between having a sensor or an actuator at a particular DOF.

## 7 Extensions for Alternative Test Setups

A final discussion should be undertaken regarding the applicability of these methods to real world experimental data. Let us, for the sake of argument, consider that the system we are working with is indeed a finite order  $N^{\text{th}}$  DOF system. By conducting a test with  $r$  actuators and measuring the response of the system at  $m$  DOFs, and assuming that we have identified  $N_M$  number of vibration modes, the following scenarios are possible:

$$(i) \quad N = N_M = (m + r - 1) = \begin{cases} m = N = N_M, r = 1 \\ r = N = N_M, m = 1 \\ m < N = N_M, r < N = N_M \end{cases}$$

$$\begin{aligned}
& \text{(ii) } N > N_M = (m + r - 1) \begin{cases} m = N_M, r = 1 \\ r = N_M, m = 1 \\ m < N_M, r < N_M \end{cases} \\
& \text{(iii) } N = N_M > (m + r - 1) \\
& \text{(iv) } N > N_M > (m + r - 1) \\
& \text{(v) } N \geq (m + r - 1) > N_M
\end{aligned}$$

With regard to case (i), all the algorithms presented in this paper can exactly address the case of a full ( $m = N$ ) set of sensors (one sensor at each DOF), while only Algorithm 2 and 3 can be used in the case of a full ( $r = N$ ) set of actuators. For an incomplete set of sensors and actuators, but such that  $m + r = N + 1$ , only Algorithm 3 can handle such a solution. This is an important consideration. In fact, multiple point actuation ( $m > 1$ ) is quite common in experimental setting, although expensive and difficult to implement, and is not always associated with co-located sensors. In civil engineering applications, multiple point actuation in bridges is quite common: as an example, the Vincent-Thomas bridge in Long-Beach (CA), one of the best instrumented bridges in the world, has 26 accelerometers deployed along its length, 10 of which are at foundation locations (input locations). Hence, having the possibility of accounting for information at actuator points, in addition to that at sensor locations, is quite relevant.

For case (ii), if the required sensor-actuator configurations are satisfied as in the above case (i), all the three algorithms provide approximate solutions to the second-order system matrices. This approximation is the result of modal truncation. In fact, the data may contain more modes than those identified and the omission of these "residual" modes can affect the estimates of the physical parameters. Clearly the errors will be more significant if there is substantial coupling between the identified and the residual modes. However, this case should be seen as a limitation not of these three algorithms but rather of the identification algorithm used for the initial first-order representation of the system. In fact, in this case the initially identified state-space formulation would not contain some of the structural modes, either because such modes are not sufficiently excited or because of wrong positioning of sensors and actuators, and this would make it impossible for the presented algorithms to retrieve those missing modes.

With regard to cases (iii), (iv) and (v), neither of the three algorithms can address these cases directly, although some solutions are possible only with appropriate modifications. It should be mentioned that, while either Algorithm 1 or Algorithm 2 in their current state cannot be employed to tackle these cases, Algorithm 3 seems to be the most flexible one in terms of incorporating such extensions. In fact, for a test setup as in case (iii) and for a generally (proportional and non-proportional) damped system, Algorithm 3 could be directly applied to estimate the values of the complex modes at both sensor and actuator locations, using eqs.(62) and (63), and consequently to obtain reduced order mass, damping and stiffness matrices of the system (Luş et al., 2002b). Such reduced-order mass and stiffness matrices will be identical to the “statically condensed” versions of the full-order mass and stiffness matrices, respectively, one would obtain by considering the DOFs with either a sensor or an actuator as “independent” while treating the remaining DOFs as “dependent” and hence condensing such DOFs. Moreover, in the case of a block diagonal mass matrix, Algorithm 3 is capable of providing the exact partitions of the mass and damping matrices corresponding to the instrumented DOFs. Algorithm 1 can also provide reduced order mass, damping and stiffness matrices in case (iii), analogous to the system matrices produced by the Guyan reduction. However, in the case of the number of sensor being less than the number of identified modes, the corrective transformation for obtaining the undamped modal properties from a damped first order realization must be solved iteratively using a two-stage Sequential Quadratic Programming procedure (Alvin, 1993). In both cases, these reduced-order systems, although obtained from a “complete” first-order system realization, are not dynamically equivalent to the real system in the sense that their eigenvalues and corresponding eigenvectors are not directly correlated to those of the real structure. However, thanks to their physical interpretation, such reduced-order representations can provide useful information with regard to damage assessment and health monitoring of structures. For the case of Algorithm 2, the modifications that need to be imposed would be so complex that it would be equivalent to the alteration of the whole algorithm. Cases (iv) and (v) are probably more challenging in the sense that the reduced order physical matrices that may be obtained may not necessarily allow physical interpretations. Therefore, it may be preferable in these cases to address the problem of estimating the modal parameters (i.e. undamped natural frequencies and mode shapes) rather than the problem of estimating the physical mass, damping and stiffness matrices (Alvin, 1993). Alternative

formulations to address the last three cases have not been thoroughly addressed in the literature and need further investigation.

## 8 Conclusions

In this article, the authors have laid down the theoretical framework for the proposed two-part study, and investigated various algorithms that have been proposed for obtaining physical parameters of mechanical systems using identified state space realizations. It has been shown that the problem under investigation can be successfully articulated as the identification of some transformation matrices, which are used to transform the first order state space equations (that are, in general, written in some arbitrary basis) to physically meaningful coordinates. The framework presented in this work has provided some common ground between the seemingly different approaches, and hopefully, has provided a better understanding of the underlying theory.

In order to evaluate the performances of the algorithms discussed herein, discussions regarding algorithmic complexity, sensitivity to perturbations, and various other concerns will be addressed in detail in the companion paper of this two-part study.

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